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Lars P. Metzger[1]

# Alliance Formation in Contests with Incomplete Information

## Abstract

*This paper studies a contest in which players with unobservable types may form an alliance in a pre-stage of the game to join their forces and compete for a prize. We characterize the pure strategy equilibria of this game of incomplete information. We show that if the formation of an alliance is voluntary, players do not reveal private information in the process of alliance formation in any equilibrium. In this case there exists a pooling equilibrium without alliances with a unique effort choice in the contest and there exist equilibria in which all types prefer to form an alliance. If the formation of an alliance can be enforced by one player with positive probability there exists an equilibrium in which only the low types prefer to form an alliance.*

*JEL Classification: C72, D72, D74, D82*

*Keywords: Alliance formation; contest; incomplete information; free-riding; signalling*

*February 2015*

*1 Lars P. Metzger, TU Dortmund.* *– The author is grateful for comments received by the audiences at the SING meeting at Krakau 2014 and UECE Lisbon Meetings on Game Theory and Application 2014, and for the admission to present this paper at the 2015 Public Choice Society Conference in San Antonio, and for invaluable comments of Wolfgang Leininger, Jörg Franke and Florian Morath.* *– All correspondence to: Lars P. Metzger, TU Dortmund, Department of Economics, 44221 Dortmund, Germany, e-mail: Lars.Metzger@tu-dortmund.de*

# 1 Introduction

The central element of the model presented in this paper is a rent seeking game as introduced by Tullock (1980). Jia, Skaperdas, and Vaidya (2013) argue that the Tullock-contest success function is well motivated by an axiomatic foundation, by stochastic functional forms or by microfounded search models. Without the simple structure of the Tullock-contest success function the model studied in the paper at hand does not seem to be tractable. There is a large number of contributions to the literature on rent seeking games with asymmetric valuations[1] and a smaller number number of contributions in which individual valuations are private information, see for example Hurley and Shogren (1998) for one sided incomplete information. Malueg and Yates (2004) study a model in which the prior of the private values is potentially correlated. Ewerhart (2010) fully characterizes the Bayesian equilibrium for symmetric rent-seeking contests with independent private valuations and Ewerhart (2014) shows uniqueness of the equilibrium for continuous types. Fey (2008) and Wasser (2013) introduce informational asymmetries on the costs of effort provision. In Morath and Münster (2013) players may acquire information on their valuations, and opponents observe whether or not information was acquired but do not learn the information. The central question which is studied in the paper which I offer is in how far agents truthfully reveal their private information in a pre-stage and in which way this influences behavior in the subsequent contest. Katsenos (2009) studies agents who send a signal on their private information before entering the contest. He shows that a separation of types in equilibrium is only possible, if opponents are weak with an a priori high probability. In this paper we allow for any probability on the set of types, our results do not depend on this probability. Zhang (2008) and Münster (2009) study a repeated contest in which the level of effort provision in the first stage reveals information on the private valuation in the second stage. Münster (2009) finds that high ability contestants aim at concealing their type by choosing a low effort in the first stage in equilibrium. Skaperdas (1998) studies a model of alliance formation between three players who compete for one prize.[2] In his model players have heterogeneous and observable types. Skaperdas gives sufficient conditions for the contest success function such that only 'weak' types respectively only 'strong' types voluntarily form an alliance. Herbst, Konrad, and Morath (2015) test a similar version of this model in an experimental

[1] See Hillman and Riley (1989), Hirshleifer (1989), Hurley (1998), Suen (1989) to name only a small subset.

[2] Skaperdas (1998) also studies multiple prize contests, which are not covered by the paper at hand.

setup but restrict all players to have the same valuation. Yet their experimental findings indicate that agents have some underlying and unobservable type which both influences the level of effort provision and causes a selection effect in the first stage of the game. The model which we present in the next section assumes an underlying *unobservable* type and analyzes the sequential Bayesian game. We show that if alliance formation is voluntary, type contingent alliance-formation is strictly unstable for a standard contest success function. In contrast the setup of Skaperdas (1998) implies that alliance formation is weakly stable for both weak and strong players, if the standard success function is applied. If the formation of an alliance can be enforced with some probability, as in Herbst, Konrad, and Morath (2015), we derive a lower bound for the strong type such that their empirical finding can be supported by an equilibrium.

# 2 Model

Assume that player $i = 1, 2, 3$ is a risk neutral expected utility maximizer and that $i$'s preferences for a lottery that allocates one prize to $i$ with probability $p \in [0, 1]$ is represented by the function

$$u_i(p, v_i) = p \cdot v_i, \; i = 1, 2, 3$$

for $v_i \in \{1, \bar{v}\}$, $\bar{v} \geq 1$. One motive for $\bar{v} > 1$ could be the 'joy of winning'[3], where we would regard $\bar{v}$ close to one. We characterize a player with $v_i = 1$ as 'weak' and a player with $v_i = \bar{v}$ as 'strong'. The valuation $v_i$ is known only to player $i$. The commonly known independent and identical a priori probability of receiving $v_i = 1$ is denoted by $q \in (0, 1)$.

The game consists of two stages; in the first stage players one and two may publicly declare whether or not they are willing to form an alliance. Player $i$ declares $a_i = yes$, if he prefers to form an alliance and $a_i = no$ otherwise. If no player prefers to form an alliance, no alliance forms. If one player prefers and the other player does not prefer to form an alliance, the alliance forms with probability $\gamma \in [0, 1]$.[4] If both players prefer to form an alliance, the alliance forms with certainty. After stage 1 all players observe whether or not players one and two prefer an alliance and whether or not the alliance actually forms. We denote the event of alliance formation by $\mathcal{A} = A$ and its complement by $\mathcal{A} = \neg A$. The publicly available information of stage 1 is captured by $a = (a_1, a_2, \mathcal{A})$. In the second stage the contest takes place

[3]See Sheremeta (2010) and the literature cited therein for several experiments that support the hypothesis of non-monetary utility of winning.

[4]In Skaperdas (1998), $\gamma = 0$, while in Herbst, Konrad, and Morath (2015), $\gamma = \frac{1}{2}$.

and each player $i$ simultaneously chooses a number $e_i \geq 0$. The rules of the contest depend on whether players one and two form an alliance or not.

## Payoffs

We consider the standard Tullock contest success function, where $p_i(e) = \frac{e_i}{\sum_j e_j}$ for $e : \sum_j e_j > 0$ and $p_i(e) = \frac{1}{3}$ if $\sum_j e_j = 0$. The probability that player three wins the contest is given by $p_i(e)$ in any case; the probility that players one or two win the contest is given by $p_i(e)$, if there is no alliance. Then, the utility function is given by

$$u_i(e_1, e_2, e_3|\neg A) = \frac{e_i}{e_1 + e_2 + e_3} \cdot v_i - e_i \ , \ i = 1, 2, 3 \ .$$

We assume that if the alliance forms and one member of the alliance wins the contest, the prize is distributed with uniform probability among all members of the alliance.[5]

If an alliance between players one and two forms and $\sum_j e_j > 0$, players 1 and 2 expect to receive

$$u_i(e_1, e_2, e_3|A) = \frac{e_1 + e_2}{e_1 + e_2 + e_3} \cdot \frac{v_i}{2} - e_i \ , \ i = 1, 2 \ .$$

If an alliance forms but $\sum_j e_j = 0$, players 1 and 2 expect to receive

$$u_i(0|A) = \frac{1}{2} \cdot \frac{v_i}{2}, \ i = 1, 2 \text{ and } u_3(0|A) = \frac{1}{2} \cdot v_3 \ .$$

If we would model the within alliance allocation of the prize using a within alliance contest, our use of the standard Tullock contest success function would imply that the payoff functions with and without alliance would coincide. Note that a uniform within alliance assignment of the prize strictly increases the incentives to form an alliance.

## Strategies

A choice for player $i = 1, 2$ in stage 1 is a mapping

$$a_i : \{1, \bar{v}\} \rightarrow \{yes, no\} \ .$$

[5] This can be seen as an extreme interpretation of the payoff-structure in Skaperdas (1998), where the prize is allocated to a member of the alliance according to the conduction of a second contest within the winning alliance. We implicitly assume that the members collusively choose $e_i = 0$ in this second contest.

A choice for player $i = 1, 2, 3$ in stage 2 is a mapping

$$e_i : \{yes, no\}^2 \times \{\neg A, A\} \times \{1, \bar{v}\} \to \mathbb{R}_+ \ .$$

A strategy for player $i = 1, 2$ is a pair of mappings $a_i$ and $e_i$.

### Beliefs

Player $i$ believes that another player $j$ has a low valuation with probability $\mu_j$. $\mu_j$ depends on the other player's choice $a_j$ in the first stage. Clearly the beliefs of players one and two for player three equal the prior probability $q$ of having a low valuation. In a separating equilibrium the beliefs for players one and two equal zero or one, in a pooling equilibrium these beliefs equal the prior probabilities.

Denote by the vector $\tilde{e}$ the beliefs of the players over the 2nd-stage choices of the other players given $a \in \{yes, no\}^2 \times \{\neg A, A\}$ and $v \in \{1, \bar{v}\}$.

### Equilibrium

The strategies $\{(a_1, e_1), (a_2, e_2), (e_3)\}$ and beliefs $\mu$, $\tilde{e}$ are a Bayesian Nash equilibrium, if

- the beliefs are formed using Bayes' rule (if possible) given the strategies and
- the strategies maximize expected payoffs given the beliefs.

In the following section we analyze equilibria with partial information revelation (separating equilibria) and equilibria with no information revelation (pooling equilibria).

# 3 Analysis

In this section we derive the necessary conditions for optimal behavior in the second stage of the game given arbitrary beliefs. In the following sections we successively solve for different scenarios of 1st-stage behavior in pure strategies. These different scenarios allow us to restrict attention to cases in which the beliefs $\mu_i$ assume values in $\{0, q, 1\}$.

If the alliance does not form, then each player $i = 1, 2, 3$ or if the alliance does form, then only player $i = 3$ maximizes the following expected payoff function which depends on the beliefs $\mu_j(a_j),\ \mu_k(a_k)$ and $\tilde{e}_j,\ \tilde{e}_k$ for $j,k \in \{1,2,3\}, j,k \neq i,\ j \neq k$ (where $\mu_j(a_j) = q$ for $j = 3$):

$$\begin{aligned}
\mu_j(a_j) \cdot \mu_k(a_k) &\cdot \frac{e_i}{e_i + \tilde{e}_j(a, v_j = 1) + \tilde{e}_k(a, v_k = 1)} \cdot v_i \quad + \\
\mu_j(a_j) \cdot (1 - \mu_k(a_k)) &\cdot \frac{e_i}{e_i + \tilde{e}_j(a, v_j = 1) + \tilde{e}_k(a, v_k = \bar{v})} \cdot v_i \quad + \\
(1 - \mu_j(a_j)) \cdot \mu_k(a_k) &\cdot \frac{e_i}{e_i + \tilde{e}_j(a, v_j = \bar{v}) + \tilde{e}_k(a, v_k = 1)} \cdot v_i \quad + \\
(1 - \mu_j(a_j)) \cdot (1 - \mu_k(a_k)) &\cdot \frac{e_i}{e_i + \tilde{e}_j(a, v_j = \bar{v}) + \tilde{e}_k(a, v_k = \bar{v})} \cdot v_i \quad - e_i
\end{aligned}$$

The first order condition to this maximization problem is satisfied by the optimal choice $e_i(a, v_i)$ of player $i$:

$$\begin{aligned}
\mu_j(a_j) \cdot \mu_k(a_k) &\cdot \frac{\tilde{e}_j(a,v_j=1)+\tilde{e}_k(a,v_k=1)}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=1)+\tilde{e}_k(a,v_k=1))^2} \cdot v_i \quad + \\
\mu_j(a_j) \cdot (1 - \mu_k(a_k)) &\cdot \frac{\tilde{e}_j(a,v_j=1)+\tilde{e}_k(a,v_k=\bar{v})}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=1)+\tilde{e}_k(a,v_k=\bar{v}))^2} \cdot v_i \quad + \\
(1 - \mu_j(a_j)) \cdot \mu_k(a_k) &\cdot \frac{\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_k(a,v_k=1)}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_k(a,v_k=1))^2} \cdot v_i \quad + \\
(1 - \mu_j(a_j)) \cdot (1 - \mu_k(a_k)) &\cdot \frac{\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_k(a,v_k=\bar{v})}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_k(a,v_k=\bar{v}))^2} \cdot v_i \quad \leq \quad 1\ ,
\end{aligned} \tag{1}$$

where $e_i(a, v_i) = 0$ whenever the inequality is strict.

If there is an alliance, player $i = 1, 2$ maximizes (with $j = 1, 2\ j \neq i$):

$$\begin{aligned}
\mu_j(a_j) \cdot q &\cdot \frac{e_i + \tilde{e}_j(a, v_j = 1)}{e_i + \tilde{e}_j(a, v_j = 1) + \tilde{e}_3(a, v_3 = 1)} \cdot \frac{v_i}{2} \quad + \\
\mu_j(a_j) \cdot (1 - q) &\cdot \frac{e_i + \tilde{e}_j(a, v_j = 1)}{e_i + \tilde{e}_j(a, v_j = 1) + \tilde{e}_3(a, v_3 = \bar{v})} \cdot \frac{v_i}{2} \quad + \\
(1 - \mu_j(a_j)) \cdot q &\cdot \frac{e_i + \tilde{e}_j(a, v_j = \bar{v})}{e_i + \tilde{e}_j(a, v_j = \bar{v}) + \tilde{e}_3(a, v_3 = 1)} \cdot \frac{v_i}{2} \quad + \\
(1 - \mu_j(a_j)) \cdot (1 - q) &\cdot \frac{e_i + \tilde{e}_j(a, v_j = \bar{v})}{e_i + \tilde{e}_j(a, v_j = \bar{v}) + \tilde{e}_3(a, v_3 = \bar{v})} \cdot \frac{v_i}{2} \quad - e_i
\end{aligned}$$

The optimal choice $e_i(a, v_i)$ for $i = 1, 2$ satisfies

$$\begin{aligned}
\mu_j(a_j) \cdot q &\cdot \frac{\tilde{e}_3(a,v_3=1)}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=1)+\tilde{e}_3(a,v_3=1))^2} \cdot \frac{v_i}{2} \quad + \\
\mu_j(a_j) \cdot (1 - q) &\cdot \frac{\tilde{e}_3(a,v_3=\bar{v})}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=1)+\tilde{e}_3(a,v_3=\bar{v}))^2} \cdot \frac{v_i}{2} \quad + \\
(1 - \mu_j(a_j)) \cdot q &\cdot \frac{\tilde{e}_3(a,v_3=1)}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_3(a,v_3=1))^2} \cdot \frac{v_i}{2} \quad + \\
(1 - \mu_j(a_j)) \cdot (1 - q) &\cdot \frac{\tilde{e}_3(a,v_3=\bar{v})}{(e_i(a,v_i)+\tilde{e}_j(a,v_j=\bar{v})+\tilde{e}_3(a,v_3=\bar{v}))^2} \cdot \frac{v_i}{2} \quad \leq \quad 1\ ,
\end{aligned} \tag{2}$$

where $e_i(a, v_i) = 0$ whenever the inequality is strict.

Note that all players face a strictly concave maximization problem. Therefore any belief has a unique best response.

## 3.1 Separating Equilibrium

In a separating equilibrium the action $a_i$ of the first stage implies a belief $\mu_i(a_i)$ which is either equal to zero or equal to one for $i = 1, 2$. Hereby, players one and two reveal their private information after the first stage. In the second stage, all players know the valuations of players one and two and whether there is an alliance between players one and two or not. There are eight cases on which all players can condition their second stage choices:

| case | # | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| valuation of player 1 | $v_1$ | 1 | 1 | 1 | 1 | $\bar{v}$ | $\bar{v}$ | $\bar{v}$ | $\bar{v}$ |
| valuation of player 2 | $v_2$ | 1 | 1 | $\bar{v}$ | $\bar{v}$ | 1 | 1 | $\bar{v}$ | $\bar{v}$ |
| formation of an alliance | $\mathcal{A}$ | $\neg A$ | $A$ | $A$ | $\neg A$ | $A$ | $\neg A$ | $\neg A$ | $A$ |

Note that cases 2 and 4 and also cases 3 and 5 are symmetric. Note further that in symmetric pure strategy equilibria cases 0 and 1 and also cases 6 and 7 are mutually exclusive.

In each of the eight cases inequalities (1) and (2) simplify substantially and we are able to derive each of the solutions in closed form as functions of the parameters $\bar{v} > 1$ and $q \in (0, 1)$. We present these solutions in appendix A. Table 1 lists these solutions for the particular values $\bar{v} = 2$ and $q = \frac{1}{2}$.

Inequalities (1) and / or (2) imply polynomials of degree four (if binding). The key to the solution is inequality (1) for player three. If (1) is binding for $i = 3$, the solution satisfies $(e_1 + e_2 + e_3(v_3))^2 = v_3 \cdot (e_1 + e_2)$, which can be substituted in (1) or (2) for players one and two and simplifies the problem substantially.

Note further for the cases in which an alliance forms, $\mathcal{A} = A$, that the only difference in inequality (2) for players one and two is the difference in the valuations $v_i$ and $v_j$. If $v_1 \neq v_2$, inequality (2) must be strict for the player with the lower valuation and that player chooses zero effort, irrespective of the magnitude of the difference in $v_1$ and $v_2$. A tiny heterogeneity suffices that one player rides on the back of the other.

| $(v_1, v_2)$ | $\mathcal{A}$ | $e_1$ | $e_2$ | $e_3(v_3 = 1)$ | $e_3(v_3 = 2)$ | $u_1$ | $u_2$ |
|---|---|---|---|---|---|---|---|
| (1,1) | $\neg A$ | 0.1927 | 0.1927 | 0.2354 | 0.4925 | 0.0723 | 0.0723 |
| (1,1) | $A$ | 0.0482 | 0.0482 | 0.2140 | 0.3426 | 0.0843 | 0.0843 |
| (1,2) | $A$ | 0 | 0.2379 | 0.2498 | 0.4519 | 0.2082 | 0.1784 |
| (1,2) | $\neg A$ | 0.0959 | 0.4797 | 0.1831 | 0.4973 | 0.0120 | 0.5996 |
| (2,1) | $A$ | 0.2379 | 0 | 0.2498 | 0.4519 | 0.1784 | 0.2082 |
| (2,1) | $\neg A$ | 0.4797 | 0.0959 | 0.1831 | 0.4973 | 0.5996 | 0.0120 |
| (2,2) | $\neg A$ | 0.4758 | 0.4758 | 0.0239 | 0.4280 | 0.3568 | 0.3568 |
| (2,2) | $A$ | 0.1189 | 0.1189 | 0.2498 | 0.4519 | 0.2974 | 0.2974 |

Table 1: solutions of inequalities (1) and / or (2) for parameter values $\bar{v} = 2$ and $q = \frac{1}{2}$.

In all cases but the two cases in which $v_1 = v_2$ and $\mathcal{A} = A$, the solutions are unique. In the two remaining cases, table 1 lists the unique symmetric solutions. We argue why we can focus on the symmetric solutions without loss of generality in subsection (3.2).

Before we continue the analysis, suppose for a moment that players one and two mutually know their valuations at the beginning of the game. Then in any case player $i = 1, 2$ prefers to form an alliance with the other player if and only if $v_i = 1$. This observation is true for general parameter values $\bar{v} > 1$ and $q \in (0, 1)$. In subsections 3.2 and 3.3 we study whether this observation carries over to the case of incomplete information.

## 3.2 Only The Strong Stand Alone

In this subsection we analyze the following first stage-choices for players one and two:

$$\hat{a}_i(v_i) = \begin{cases} yes & \text{if } v_i = 1 \\ no & \text{if } v_i = \bar{v} \end{cases}, \ i = 1, 2$$

Given these choices all players learn the valuations $v_1$ and $v_2$ after stage 1 (and mutually know this). Therefore, $\mu_i(yes) = 1$ and $\mu_i(no) = 0$ for $i = 1, 2$.

Note that cases 0 and 7 cannot emerge. If an alliance requires consent ($\gamma = 0$), cases 2 and 4 cease to exist while if an alliance can be enforced by one player ($\gamma = 1$), cases 3 and 5 cease to exist.

In case $v_1 = 1$, $v_2 = 1$, $\hat{a} = (yes, yes, A)$ there is a continuum of mutually optimal choices. All efforts of players one and two that sum up to

$\bar{v} \cdot \left(\frac{1+q\cdot(\sqrt{\bar{v}}-1)}{1+2\cdot\bar{v}+q\cdot(\bar{v}-1)}\right)^2$ are mutually optimal.[6] Clearly, player one prefers those pairs of mutually optimal efforts, which imply a lower effort for player one. If players one and two exert asymmetric efforts, the outcome is less attractive to the player with the higher effort who then might have stronger incentives to deviate from the decision in the first stage. If we can show that symmetric effort choices in the second stage imply a deviation in the first stage, then asymmetric effort choices would all the more imply a deviation in the first stage. By no means we suggest that the symmetric case is most plausible. Consider the best pair of mutually optimal choices for player one in which he chooses $e_1 = 0$ and completely free rides on the effort of player two who bears the whole burden and chooses $e_2 = \bar{v} \cdot \left(\frac{1+q\cdot(\sqrt{\bar{v}}-1)}{1+2\cdot\bar{v}+q\cdot(\bar{v}-1)}\right)^2$. Consider now the situation $v_1 = 1,\ v_2 = \bar{v}$ and $\hat{a} = (yes, no, A)$. Here, players one and two form an alliance (against the will of player two) and again player one completely free rides on the effort choice of player two with $e_1 = 0$ and $e_2 = \bar{v} \cdot \left(\frac{1+q\cdot(\sqrt{\bar{v}}-1)}{3+q\cdot(\bar{v}-1)}\right)^2$. Observe that if $\bar{v}$ approaches 1, the mutually optimal effort choices approach the most preferred ones for player one in the situation $v_1 = 1,\ v_2 = 1,\ \hat{a} = (yes, yes, A)$.

While table 1 includes the mutually optimal effort choices and utilities along the potential equilibrium path, we need to derive the optimal effort choices and utilities after a unilateral and unexpected deviation.

### 3.2.1 Optimal Effort Choices After A Unilateral Deviation

We turn now to optimal choices of player one in stage 2 after a deviation of player one to $\tilde{a}_1(v_1 = 1) = no$ or $\tilde{a}_1(v_1 = \bar{v}) = yes$ in stage 1. In this case players two and three believe that players one and two use the strategy $\hat{a}_1$ and $\hat{a}_2$. Therefore, the actual choice $e_1$ does not equal player three's belief $\tilde{e}_1$ and therefore inequality (1) cannot be used to simplify the optimization problem as before. While it is straight forward to show that player one's maximization problem is strictly concave and therefore has a unique maximizer, we cannot provide a solution in closed form. Instead, we derive the solutions numerically. For this reason we focus on the specification $q = \frac{1}{2}$ and $\bar{v} = 2$. The reader can reproduce these derivations by using the supplementary files, which also allow for general parameter values. The derivation is illustrated in appendix B. In section 3.2.3 we argue that these results hold for a general range of the parameter $\bar{v}$.

[6] See appendix A for all details.

Table 2 lists the utilities of player one derived from optimal choices of player one after a deviation from $\hat{a}_1$ to $\tilde{a}_1$ given that players two and three behave in the belief that player one uses $\hat{a}_1$ instead of $\tilde{a}_1$.

| $(v_1, v_2)$ | $(\tilde{a}_1, \hat{a}_2, \mathcal{A})$ | $u_1(\hat{e}_1, e_{-1}\|v_1)$ |
|---|---|---|
| $(1, 1)$ | $(no, yes, A)$ | 0.0183 |
| $(1, 1)$ | $(no, yes, \neg A)$ | 0.1357 |
| $(1, 2)$ | $(no, no, \neg A)$ | 0.0366 |
| $(2, 1)$ | $(yes, yes, A)$ | 0.2786 |
| $(2, 2)$ | $(yes, no, A)$ | 0.4163 |
| $(2, 2)$ | $(yes, no, \neg A)$ | 0.2698 |

Table 2: utilities of player one for optimal deviations

### 3.2.2 Alliance Formation in Stage 1

The following game tree summarizes the findings for the case in which player two uses $\hat{a}_2$ and in which players two and three believe that player one uses $\hat{a}_1$:

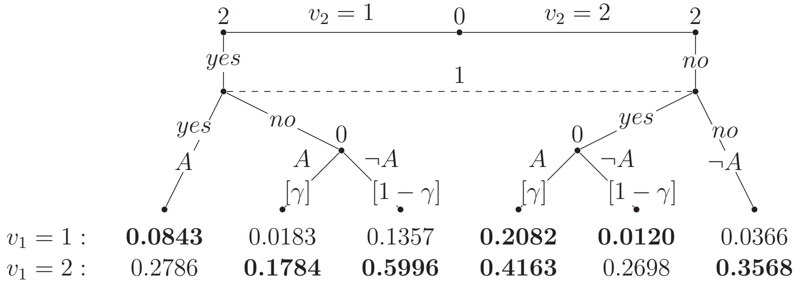


Figure 1: Reduced game tree from the perspective of player 1. The numbers are the expected utilities of player 1. Bold numbers indicate analytical derivations.

In stage 1 player one does not know $v_2$ and whether or not a unilateral offer suffices for an alliance. Figure 2 illustrates player one's expected utility derived from $\hat{a}_1$ and $\tilde{a}_1$ using the priors $q = \frac{1}{2}$ and $\gamma \in (0, 1)$.

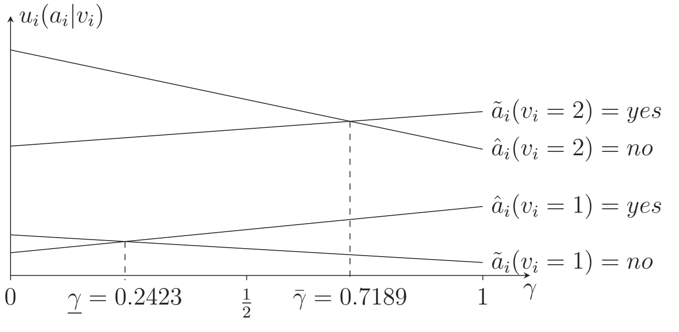


Figure 2: Expected utilities of player $i$ in stage 1

We may conclude that player $i = 1, 2$ with $v_i = 1$ does not have an incentive to deviate to $\tilde{a}_i(v_i = 1) = no$, if $\gamma > \underline{\gamma} = 0.2423$ and that player $i = 1, 2$ with $v_i = 2$ does not have an incentive to deviate to $\tilde{a}_i(v_i = 2) = yes$, if $\gamma < \bar{\gamma} = 0.7189$.

**Proposition 1 (Separating Equilibrium)** *For $\bar{v} = 2$ and $q = \frac{1}{2}$, there is an open set $(\underline{\gamma}, \bar{\gamma}) \subset (0, 1)$ with $\frac{1}{2} \in (\underline{\gamma}, \bar{\gamma})$ such that the choice to offer an alliance if and only if $v_i = 1$ is part of a strategy of a separating equilibrium if and only if $\gamma \in (\underline{\gamma}, \bar{\gamma})$. More precisely:*

$$\hat{a}_i(v_i) = \begin{cases} yes & \text{if } v_i = 1 \\ no & \text{if } v_i = 2 \end{cases} \text{ is part of a separating equilibrium}$$
$$\Leftrightarrow$$
$$\gamma \in (\underline{\gamma}, \bar{\gamma}) \approx (0.2423, 0.7189).$$

*Appendices A and B summarize the optimal choices in the second stage.*

In this section we derive optimal behavior for the parameter specification $\bar{v} = 2$ and $q = \frac{1}{2}$. We derive equilibrium choices as closed form solutions in the parameters $\bar{v}$, $q$ and $\gamma$. We derive the choices in the second stage that follow deviation choices in the first stage numerically. The reader can verify these numerical results using very simple means at any precision that is desirable. We present the results with a four digit precision. The next subsection applies this methodology for a broader specification of the parameter $\bar{v}$.

### 3.2.3 Optimal Alliance Formation for $\bar{v} \in [1,3]$

By repeating the procedure for any $\bar{v} \in (1,3]$ (holding $q = \frac{1}{2}$ fixed) we can illustrate the dependence of $\bar{\gamma}$ and $\underline{\gamma}$, the upper and lower bound for $\gamma$ on $\bar{v}$ in figure 3:[7]

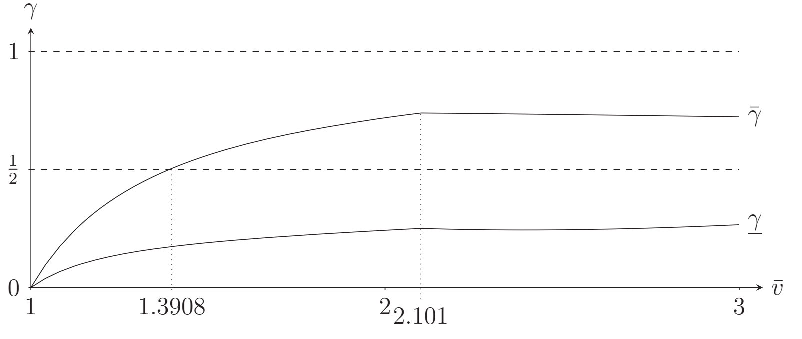


Figure 3: For each parameter $\bar{v} \in [1,3]$ (holding $q = \frac{1}{2}$ fixed) the two lines depict the lower and upper bound of the interval $(\underline{\gamma}, \bar{\gamma})$ for which the choice $a_i = yes \Leftrightarrow v_i = 1$ is part of a separating equilibrium.

Figure 3 reveals that $\underline{\gamma}, \bar{\gamma} \underset{\bar{v}\to 1}{\longrightarrow} 0$ and that $\underline{\gamma} > 0$ for all $\bar{v} \in (1,3]$, which results in the following observations: If $\gamma = \frac{1}{2}$ (as in Herbst, Konrad, and Morath (2015)), the choice $\hat{a}_i$ can be supported as part of an equilibrium strategy if and only if the unobservable high valuation is large enough ($\bar{v} \geq 1.3908$). If there is only little heterogeneity ($\bar{v}$ is close to 1), the range for admissable values for $\gamma$ collapses to a tiny interval close to zero with the interpretation that an alliance forms only if both players agree to form an alliance.

We compare these results to Skaperdas (1998) in section 4.

## 3.3 Only The Weak Stand Alone

In this subsection we analyze the consequences of the choice

$$\tilde{a}_i(v_i) = \begin{cases} no & , \text{if } v_i = 1 \\ yes & , \text{if } v_i = \bar{v} \end{cases}$$

[7]There is a kink at $\bar{v} = 2.101$, because in case #6 player three chooses $e_3(v_3 = 1) = 0$ for all $\bar{v} \geq 2.101$.

in stage 1. We show that this choice in the first stage of the game cannot be a part of a separating equilibrium. As in subsection 3.2, players one and two reveal their valuations after stage 1 such that we can conduct backwards induction. The beliefs collapse to $\mu_i(a_i = yes) = 0$ and $\mu_i(a_i = no) = 1$ for $i = 1, 2$. Appendix A lists the mutually optimal effort choices and utilities. We then assume that player one unilaterally deviates in stage 1 and derive player one's best responses to the choices of players two and three who believe that player one chooses according to $\tilde{a}_1$ in stage 1. For the same reasons as in section 3.2 we have to derive these best responses numerically for specific values of the parameters $\bar{v}$ and $q$. Finally, we show that for each $\gamma \in [0, 1]$ there is a type $v_i \in \{1, 2\}$ such that player $i = 1, 2$ has an incentive to deviate from the strategy $\tilde{a}_i$. We argue that this result generalizes to any $\bar{v} \in (1, 3]$.

### 3.3.1 Optimal Effort Choices After A Unilateral Deviation

We turn now to optimal choices of player one after a unilateral deviation of player one from $\tilde{a}_1$ to $\hat{a}_1$, where players two and three choose their efforts in the belief that player one follows $\tilde{a}_1$. As in section 3.2 we need to rely on numerical methods to derive the best responses of player one and for this reason we need to specify the values of the parameters $\bar{v} = 2$ and $q = \frac{1}{2}$. We list the maximal utilities for player one in table 3. The reader can verify these numbers by using the information provided in appendix C or using these supplementary files.

| $(v_1, v_2)$ | $(\hat{a}_1, \tilde{a}_2, \mathcal{A})$ | $u_1(\hat{e}_1, e_{-1}\|v_1)$ |
|---|---|---|
| (1,1) | $(yes, no, A)$ | 0.0183 |
| (1,1) | $(yes, no, \neg A)$ | 0.1357 |
| (1,2) | $(yes, yes, A)$ | 0.1327 |
| (2,1) | $(no, no, \neg A)$ | 0.4607 |
| (2,2) | $(no, yes, A)$ | 0.4163 |
| (2,2) | $(no, yes, \neg A)$ | 0.2698 |

Table 3: utilities of player one for optimal deviations

### 3.3.2 Alliance Formation in Stage 1 for $\bar{v} = 2$ and $q = \frac{1}{2}$

The following game tree summarizes the findings for the case in which player two chooses according to $\tilde{a}_2$ and players two and three believe that player one also chooses according to $\tilde{a}_1$.

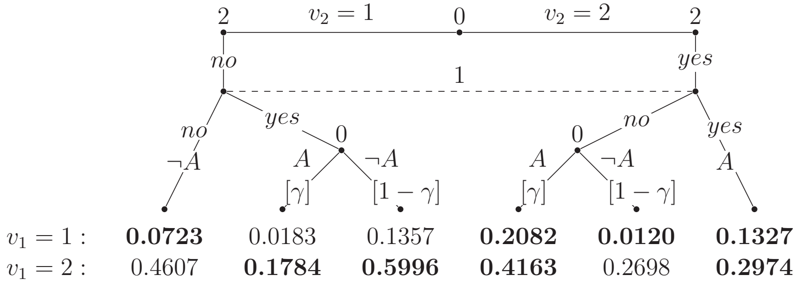


Figure 4: Reduced game tree from the perspective of player 1. The numbers are the expected utilities of player 1. Bold numbers indicate analytical derivations.

Figure 5 depicts the utility of player one with $v_1$ given that player two chooses the equilibrium candidate strategy $\tilde{a}_2$ and believes that player one also does so.

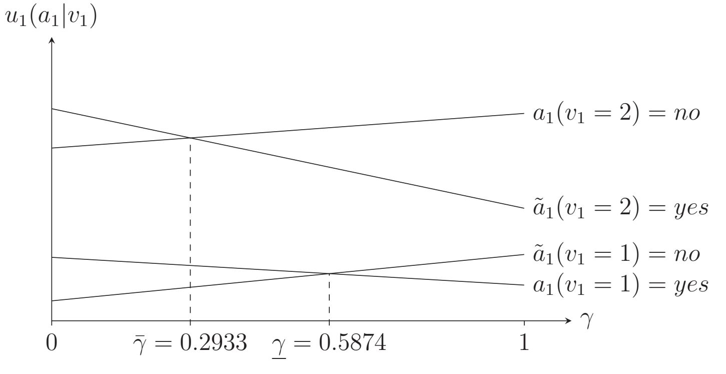


Figure 5: expected utilities of player one in stage 1

A player with $v_i = 1$ has an incentive to deviate from the equilibrium candidate strategy in the first stage, if $\gamma < \underline{\gamma}$ and a player with $v_i = 2$ has an incentive to deviate from the equilibrium candidate strategy in the first stage, if $\gamma > \bar{\gamma}$. As $\bar{\gamma} < \underline{\gamma}$, the candidate strategy $\tilde{a}_i$ of this section is not an equilibrium strategy. Proposition 2 summarizes this result:

**Proposition 2** *For any $\gamma \in [0,1]$, there is a type $v_i \in \{1,2\}$ such that player $i$ with type $v_i$ has an incentive to deviate from the equilibrium candidate choice in stage 1. The strategy* $\tilde{a}_i(v_i) = \begin{cases} no & \text{if } v_i = 1 \\ yes & \text{if } v_i = 2 \end{cases}$ *is not part of an equilibrium strategy.*

### 3.3.3 Optimal Alliance Formation for $\bar{v} \in (1,3]$

The result of the previous subsection – that $\tilde{a}_i$ is not a part of an equilibrium strategy for $\bar{v} = 2$ – generalizes to the parameter range $\bar{v} \in (1,3]$. All the calculations done for the case $\bar{v} = 2$ and $q = \frac{1}{2}$ are repeated for each $\bar{v} \in (1,3]$. Figure 6 summarizes the dependence of the lower bound $\underline{\gamma}$ and upper bound $\bar{\gamma}$ for a potential interval of probabilities $\gamma$ such that the choice $\tilde{a}_i$ would be an equilibrium choice. As there is no $\bar{v}$ such that $\underline{\gamma} < \bar{\gamma}$, the respective interval $[\underline{\gamma}, \bar{\gamma}]$ is empty for all $\bar{v} \in (1,3]$.

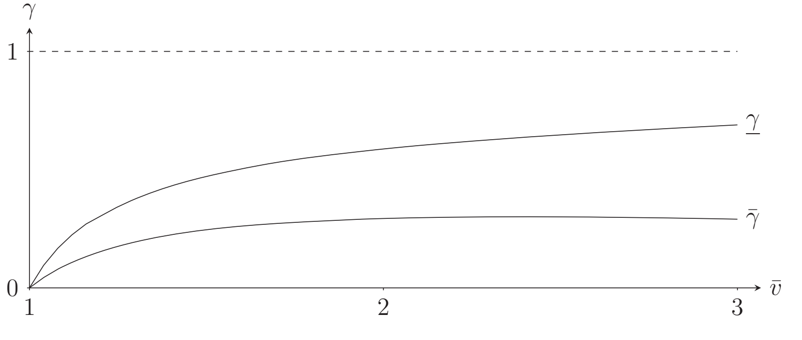


Figure 6: for any parameter value $\bar{v} > 1$ the interval $[\underline{\gamma}, \bar{\gamma}]$ is empty.

## 3.4 All Stand Alone

In this section we derive a pooling equilibrium in which players one and two reject to form an alliance – regardless of their respective valuation. In this section we allow for any probability $q \in (0,1)$ for having a valuation $v_i = 1$ and for any high valuation $\bar{v} \geq 1$.

Suppose agents choose $a_i(v_i) = no$ for $v_i \in \{1, \bar{v}\}$ and $i = 1, 2$ in the first stage of the game. Then there is no agent who receives new information about the valuations of the other players after stage 1 to update the beliefs along the unique game path that receives positive probability. Along this path there is no alliance in stage 2. In the second stage the choices of players $i = 1, 2, 3$ satisfy inequality (1) with $\mu_i(no) = q$, $i = 1, \ldots, 3$. We may exploit symmetry such that $e_i(v) =: e(v)$ for $i = 1, 2, 3$ and $v \in \{1, \bar{v}\}$, which reduces the first order conditions to:

$$q^2 \cdot \frac{1}{9 \cdot e(1)} + q \cdot (1-q) \cdot \frac{e(1) + e(\bar{v})}{(2 \cdot e(1) + e(\bar{v}))^2} + (1-q)^2 \cdot \frac{e(\bar{v})}{(e(1) + 2 \cdot e(\bar{v}))^2} \leq \frac{1}{2} \quad (3)$$

$$q^2 \cdot \frac{e(1)}{(e(\bar{v}) + 2e(1))^2} + q \cdot (1-q) \cdot \frac{e(1) + e(\bar{v})}{(e(1) + 2e(\bar{v}))^2} + (1-q)^2 \cdot \frac{1}{9e(\bar{v})} \leq \frac{1}{2\bar{v}}, \quad (4)$$

where inequality (3) is strict, if $e(1) = 0$ and (4) is strict, if $e(\bar{v}) = 0$.

**Proposition 3** *For each $q \in (0, 1)$ and $\bar{v} > 1$ there exist positive $e^*(1)$ and $e^*(\bar{v})$ that satisfy inequalities (3) and (4).*

We prove the proposition in appendix D. The proof makes use of the implicit function theorem and Brouwer's fixed point theorem.

Figure 7 depicts combinations of $e(1)$ and $e(\bar{v})$ such that conditions (3) and (4) are satisfied for parameter values $q = \frac{1}{2}$ and $\bar{v} = 2$.

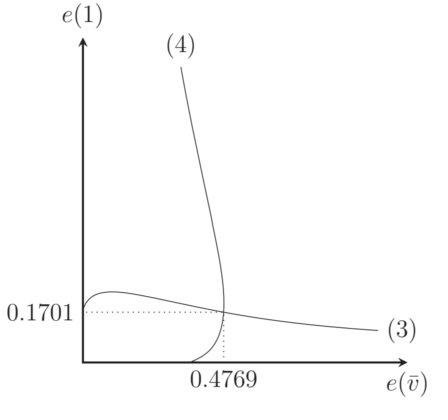


Figure 7: Condition (3) is satisfied along the curve going from west to east and condition (4) is satisfied along the curve going from south to north. Here, $q = \frac{1}{2}$ and $\bar{v} = 2$.

While we cannot obtain a solution $e^*(1|q,\bar{v})$, $e^*(\bar{v}|q,\bar{v})$ in explicit form, we can solve conditions (3) and (4) for particular values of $q$ and $\bar{v}$ numerically. For example, if $q = \frac{1}{2}$ and $\bar{v} = 2$ we have

$$e_1^*(1) = e_2^*(1) = e_3^*(1) \approx 0.1701 \text{ and } e_1^*(2) = e_2^*(2) = e_3^*(2) \approx 0.4769 \ .$$

Given these strategies, expected utilities are

$$Eu_i[e_1^*(v_1), e_2^*(v_2), e_3^*(v_3)|v_i] \approx \begin{cases} 0.0554 & \text{if } v_i = 1 \\ 0.4066 & \text{if } v_i = 2 \ . \end{cases}$$

Fortunately we do not need to know the explicit solution of equations (3) and (4) to prove the existence of a pooling equilibrium. The line of proof is as follows: any player can assure a payoff of zero by choosing $e(v_i) = 0$. As the utility functions are strictly concave any mutually optimal positive effort choices must induce a positive payoff. We specifiy (extremely pessimistic) beliefs for the off-equilibrium paths induced by at least one offer to form an alliance in stage 1 such that the maximal (believed) payoff a player can hope for is equal to zero. Given these beliefs, a deviation in the first stage is not profitable.

**Proposition 4** *For each $q \in (0,1)$ and $\bar{v} \geq 1$ there exists a Bayesian Nash equilibrium in which all players reject to form an alliance.*

Proof : Denote the payoffs given the solutions $e^*(1)$ and $e^*(\bar{v})$ for general parameters $q \in (0,1)$ and $\bar{v} > 1$ by $u^*(1)$ and $u^*(\bar{v})$. Clearly, $u*(1), u^*(\bar{v}) > 0$. Consider now a deviation of player $j \in \{1,2\}$ to $\hat{a}_j = yes$. Given that $a_i = no$ for $i = 1,2,\ i \neq j$, this deviation may or may not enforce an alliance between players one and two. If $a_j = no$ is part of the equilibrium, then Bayes' rule does not apply in the information set with $\hat{a}_j = yes$ and *any* beliefs $\tilde{e}$ can be used as equilibrium beliefs. Assume that if the deviation does not enforce an alliance, then player $j$ believes that the other players choose $\tilde{e}_k(1) = e_k^*(1)$ and $\tilde{e}_k(\bar{v}) = e_k^*(\bar{v})$ for $k = 1,2,3\ k \neq j$ just as there were no deviation. Assume that if the deviation enforces an alliance, then the deviating player $j$ believes that player $i$ chooses $\tilde{e}_i = 0$ and that player three chooses $\tilde{e}_3 = \bar{v}$. Player $i$'s belief for player $j$'s behavior can be arbitrary. Given these beliefs, it is optimal for player $j$ to choose $e_j(1) = e_j^*(1)$ and $e_j(\bar{v}) = e_j^*(\bar{v})$ if the alliance does not form, implying the payoffs $u^*(1)$ and $u^*(\bar{v})$. If the alliance does form (which would happen with probability $\gamma \in [0,1]$) it is optimal for player $j$ to choose $\hat{e}_j(1) = \hat{e}_j(\bar{v}) = 0$, which implies utility zero. Therefore, given these beliefs it does not pay to deviate from the equilibrium strategy which demands that $a_i(v_i) = no$ for $i = 1,2$ and $v_i \in \{1,\bar{v}\}$. □

The pooling equilibrium which we have just proven to exist might seem unsatisfactory because its construction relies on maximally pessimistic beliefs. The next propostion states that any beliefs that support the choices $(no, no)$ in equilibrium induce the same behavior in the second stage along the equilibrium path.

**Theorem 1** *For each $q \in (0,1)$ and $\bar{v} > 1$ the choices in stage 2 along the equilbrium path are unique for* any *Bayesian Nash equilibrium in which all players reject to form an alliance in stage 1.*

Appendix D proves this statement. The main argument of the proof is that the best respond curve for $v_i = 1$ must hit the best respond curve for $v_i = \bar{v}$ from the same side in any fixed point. If there were several fixed points, this would not be possible.

## 3.5 All Stand Together

In this section we analyze the first-stage choices $a_i(v_i) \equiv yes$ for $i = 1, 2$ and $v_i = 1, \bar{v}$. As in section 3.4 players cannot update their beliefs after stage 1 and $\mu_i = q$ for $i = 1, 2, 3$. We know from section 3.2 that there may by multiple effort choices which are mutually optimal, if players one and two form an alliance in stage 1. Therefore, in contrast to section 3.4 we do not have unique equilibrium choices in stage 2 and we can only prove existence of an equilibrium. In the following we show that there exists a symmetric equilibrium. For the choices of players one and two symmetry implies $e_1(v) = e_2(v) =: e(v)$ $, v \in \{1, \bar{v}\}$. $e(1)$ and $e(\bar{v})$ satisfy (2) which simplifies to

$$\begin{array}{rcccl} q^2 \cdot \frac{e_3(1)}{(2e(1)+e_3(1))^2} & + & q \cdot (1-q) \cdot \frac{e_3(\bar{v})}{(2e(1)+e_3(\bar{v}))^2} & + & \\ (1-q) \cdot q \cdot \frac{e_3(1)}{(e(1)+e(\bar{v})+e_3(1))^2} & + & (1-q)^2 \cdot \frac{e_3(\bar{v})}{(e(1)+e(\bar{v})+e_3(\bar{v}))^2} & \leq & 2 \end{array} \tag{5}$$

and

$$\begin{array}{rcccl} q^2 \cdot \frac{e_3(1)}{(e(\bar{v})+e(1)+e_3(1))^2} & + & q \cdot (1-q) \cdot \frac{e_3(\bar{v})}{(e(\bar{v})+e(1)+e_3(\bar{v}))^2} & + & \\ (1-q) \cdot q \cdot \frac{e_3(1)}{(2e(\bar{v})+e_3(1))^2} & + & (1-q)^2 \cdot \frac{e_3(\bar{v})}{(2e(\bar{v})+e_3(\bar{v}))^2} & \leq & \frac{2}{\bar{v}}\ , \end{array} \tag{6}$$

where $e(1) = 0$, if (5) is strict and $e(\bar{v}) = 0$, if (6) is strict.

The choices of player three satisfy (1) which simplify to

$$q^2 \cdot \frac{e(1)}{(e_3(1)+2e(1))^2} + q\cdot(1-q)\cdot\frac{e(1)+e(\bar v)}{(e_3(1)+e(1)+e(\bar v))^2} + (1-q)^2\cdot\frac{e(\bar v)}{(e_3(1)+2e(\bar v))^2} \le \frac{1}{2} \tag{7}$$

and

$$q^2 \cdot \frac{e(1)}{(e_3(\bar v)+2e(1))^2} + q\cdot(1-q)\cdot\frac{e(1)+e(\bar v)}{(e_3(\bar v)+e(1)+e(\bar v))^2} + (1-q)^2\cdot\frac{e(\bar v)}{(e_3(\bar v)+2e(\bar v))^2} \le \frac{1}{2\cdot\bar v}, \tag{8}$$

where $e_3(1) = 0$, if inequality (7) is strict and $e_3(\bar v) = 0$, if inequality (8) is strict.

If $q = \frac{1}{2}$, we can derive a symmetric equilibrium numerically with positive effort choices for each $\bar v \ge 1$. Figure 8 illustrates the effort choices along the equilibrium path.

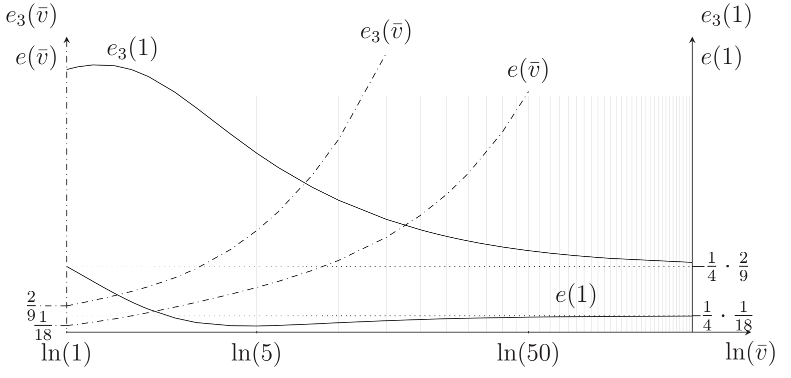


Figure 8: Symmetric equilibrium effort choices $e(v)$ of players one and two and $e_3(v)$ of player three for the valuations $v \in \{1, \bar v\}$. The right vertical axis for $v = 1$ has a ten-fold scale of the left vertical axis for $v = \bar v$.

In appendix E we argue that for any $q$ and $\bar v \ge 1$ at least three inequalities must be binding in any symmetric equilibrium and that the choices for a given high valuation $\bar v$ tend to infinity as $\bar v \to \infty$. Hereby it is straight forward to see that if (7) is binding for large enough $\bar v$, then (5) must be strict, if $q$ is close enough to zero. Therefore in contrast to section 3.4 we cannot simply treat conditions (5) to (8) as a system of equations. Robinson (1991) provides an instruction of how the implicit function theorem can be applied, if the underlying function is 'nonsmooth'. We extract the essential step for our special setting of his general analysis in lemma 1.[8] For this purpose we need to define the projection of some vector $u \in \mathbb{R}^4$ on the non-negative reals: $u^+ = (\max\{u_1, 0\}, \ldots, \max\{u_4, 0\})$. Suppose there is a function $F(u, y)$ that represents four inequalities for four choices $y \in \mathbb{R}^4$ and the optimal reactions $u \in \mathbb{R}^4$ to $y$. Lemma 1 states that any root of the auxiliar function $F(u^+, y) + u^+ - u$ implys a solution to the inequalities represented by $F(\cdot, \cdot)$. This is the key argument which allows us to use the implicit function theorem as in the proof of lemma 3 in appendix D.

**Lemma 1** *If for a given function* $F : \mathbb{R}^4 \times \mathbb{R}^4_+ \to \mathbb{R}^4$ *and* $y \in \mathbb{R}^4_+$ *the vector* $u \in \mathbb{R}^4$ *solves*

$$F(u^+, y) + u^+ - u = 0 \ ,$$

*then* $-F(u^+, y) \in \mathbb{R}^4_+$ *with* $F_i(u^+, y) < 0 \Rightarrow u_i^+ = 0$.

PROOF : Suppose $F_i(u^+, y) > 0$ for some $i = 1, \ldots, 4$. Then $u_i > u_i^+$, a contradiction. Hence $-F(u^+, y) \in \mathbb{R}^4_+$. Suppose $F_i(u^+, y) < 0$ for some $i = 1, \ldots, 4$. Then $u_i < u_i^+$, which implies $u_i^+ = 0$. □

Proposition 5 states that there exist symmetric effort choices which are mutually optimal given that players one and two of any type $v \in \{1, \bar{v}\}$ form an alliance in stage 1. As in the case of the separating equilibrium 'only the strong stand alone' in which players one with a low valuation form an alliance, there are also asymmetric effort choices in stage two which are mutually optimal. Therefore, the equilibrium which we derive in this section is not unique.

**Proposition 5**

*i There exists a vector* $e^* = (e^*(1), e_3^*(1), e^*(\bar{v}), e_3^*(\bar{v})) \in \mathbb{R}^4_+$ *wich satisfies inequalities* (5) *to* (8).

*Any such vector satisfies*

*ii* $e^*(1) + e_3^*(1) > 0,\ e^*(1) \le e^*(\bar{v}),\ e_3^*(1) \le e_3^*(\bar{v}),\ e^*(\bar{v}), e_3^*(\bar{v}) > 0$ *and*

*iii* $\lim_{\bar{v}\to\infty} e^*(1) = q^2 \cdot \frac{1}{18},\ \lim_{\bar{v}\to\infty} e_3^*(1) = q^2 \cdot \frac{2}{9}$ *and* $e^*(\bar{v}), e_3^*(\bar{v}) \underset{\bar{v}\to\infty}{\longrightarrow} \infty$ *and*

[8] Lemma 1 provides a formalization of one direction of the respective verbal statement in Robinson (1991): '...then solving (4.2) is equivalent to finding a solution of (4.5)...', p.304 within our special framework.

*iv* $\lim_{\bar{v}\to 1} e^*(1), e^*(\bar{v}) = \frac{1}{18}, \ \lim_{\bar{v}\to\infty} e_3^*(1), e_3^*(\bar{v}) = \frac{2}{9}$ .

The proof in appendix E firstly proves $\alpha$-versions of statements $i$ to $ii$ for some small but positive $\alpha$ and then uses $\alpha - ii$ to show that $i$ is also satisfied for $\alpha = 0$. Statements $iii$ and $iv$ follow directly.

Any strictly positive equilibrium effort choice induces a positive equilibrium payoff, because the utility function is concave in the own effort-level.

Given that there exist mutually optimal effort choices in stage 2 after all types of player one and two choose *yes* in stage 1 we need to construct beliefs that support these choices in stage 1. As in the equilibrium 'all stand alone', the following (extremely pessimistic) beliefs induce zero payoffs off the equilibrium path: if at least one player chooses *no* in stage 1, players one and two believe that player three chooses $e_3 = \bar{v}$. The best responses of players one and two to this belief is to choose $e_i = 0$ in stage 2, $i = 1, 2$, which results in a payoff of zero.

**Corollary 1** *There exists a pooling equilibrium in which players one and two choose yes in stage 1.*

# 4 A Comparison to Skaperdas (1998)

The model at hand differs to the model of Skaperdas (1998) with respect to the following aspects:

1. We use the standard Tullock contest success function [CSF] whereas Skaperdas (1998) allows for more general functions.

2. Skaperdas (1998) allows only for voluntary alliance formation whereas we allow for a probabilistic unilateral enforcement of an alliance which we capture by the parameter $\gamma \in [0, 1]$.

3. In Skaperdas (1998) the members of the winning alliance need to conduct a second contest to allocate the good whereas the paper at hand implicitly assumes that the members of the winning alliance perfectly collude in this second contest and allocate the good randomly.

4. In the paper at hand the valuations of the opponents are unknown to all players whereas Skaperdas (1998) assumes complete information.

Firstly we may compare our results only to those results of Skaperdas (1998) which apply for the standard Tullock CSF and secondly which are valid for

$\gamma = 0$. Thirdly we need to be aware that our model is much more in favor towards the formation of alliances than the model of Skaperdas (1998). With 4. we are able to identify the implication of incomplete information in the model of alliance formation.

According to Proposition 1 of Skaperdas (1998) there exist two players who voluntary form an alliance in the case of the standard Tullock CSF. In Proposition 2, Skaperdas (1998) provides sufficient conditions on the CSF for the identification of the types who voluntary form an alliance. The standard Tullock CSF does not satisfy these sufficient conditions; in this trivial case any type combination would voluntarily form an alliance under complete information and with a competitive secondary contest. In our model with a collusive secondary contest, incomplete information, $\gamma = 0$ and the standard Tullock CSF Proposition 1 is still valid: there exist two players who voluntarily form an alliance. In contrast to Proposition 2, the alliance is formed by any combination of types. In particular we argue that there is no voluntary alliance formation with type revelation, where only the weak players or only the strong players form an alliance. The unique voluntary alliance is the one in which players prefer to form an alliance regardless of their respective type, which can only be support by extremely pessimistic beliefs.

# 5 Conclusions

In this paper we analyze a three player game with two stages in which firstly two of the three players may opt to form an alliance and secondly all players compete in a standard Tullock (1980) contest. This model is motivated by the experiments described in Herbst, Konrad, and Morath (2015). Here, we analyze a variant in which all players have private valuations for the prize. In the experiments alliances can be enforced by a single player with probability $\frac{1}{2}$, or in my variant with some probability $\gamma \in [0, 1]$. We focus on a situation in which players use the first-stage-decision of group formation as an informative signal on the unobservable valuations. In particular we characterize a separating equilibrium in which players with a low valuation offer to form an alliance, players with a high valuation reject to form an alliance, update their beliefs accordingly and choose optimal levels of effort-provision in the second stage. We show that this equilibrium exists only if the parameter $\gamma$ is close enough to $\frac{1}{2}$. If $\gamma = 0$, any first stage equilibrium decision is uninformative regarding the types of the players. We show that a strategy in which only high types form an alliance is not part of any equilibrium for any parameter values. We proove existence of the two pooling equilibria in which either all types form an alliance or reject to form an alliance. We show that all equilibria in which no player-type prefers to form an alliance induce the same choices along the equilibrium path.

# Appendices

## A Optimal Efforts For Known $(v_1, v_2)$, And $\mathcal{A}$

We present the optimal choices $e_1(v_1, v_2, \mathcal{A}), e_2(v_1, v_2, \mathcal{A})$ and $e_3(v_1, v_2, v_3, \mathcal{A})$ that satisfy conditions (1) and / or (2) and the corresponding utilities of players one and two.

Define the following constants:

$$\begin{aligned}
C_1 &= \bar{v} \cdot \left( \frac{1 + q \cdot (\sqrt{\bar{v}} - 1)}{1 + 2 \cdot \bar{v} + q \cdot (\bar{v} - 1)} \right)^2 \\
C_2 &= \bar{v} \cdot \left( \frac{1 + q \cdot (\sqrt{\bar{v}} - 1)}{3 + q \cdot (\bar{v} - 1)} \right)^2 \\
C_3 &= 2 \cdot \bar{v} \cdot \left( \frac{1 + q(\sqrt{\bar{v}} - 1)}{2 + \bar{v} + q \cdot (\bar{v} - 1)} \right)^2 \\
\tilde{C}_3 &= \bar{v} \cdot \left( \frac{1 + q \cdot (\sqrt{\bar{v}} - 1)}{2 + q \cdot (\bar{v} - 1)} \right)^2
\end{aligned}$$

### A.0 $(v_1, v_2) = (1, 1)$ $\mathcal{A} = \neg A$

Condition (1) is uniquely satisfied by

$$\begin{aligned}
e_1 = e2 &= 2 \cdot C_1 \\
e_3(v_3 = 1) &= 2 \cdot C_1 \cdot \frac{\bar{v} + (1 - q) \cdot (\sqrt{\bar{v}} - 1)^2}{1 + q(\sqrt{\bar{v}} - 1)} \cdot \frac{1}{\sqrt{\bar{v}}} \\
e_3(v_3 = \bar{v}) &= 2 \cdot C_1 \cdot \frac{2 \cdot \bar{v} - 1 + q \cdot (\sqrt{\bar{v}} - 1)^2}{1 + q \cdot (\sqrt{\bar{v}} - 1)} \\
u_1 = u_2 &= C_1 \cdot \frac{1 + q \cdot (\bar{v} - 1)}{\bar{v}}, \; i = 1, 2 \; ,
\end{aligned}$$

## A.1 $(v_1, v_2) = (1, 1)$ , $\mathcal{A} = A$

Conditions (1) and (2) are satisfied by all $e_1, e_2 \geq 0$ such that

$$\begin{aligned} e_1 + e_2 &= C_1 \\ e_3(v_3 = 1) &= C_1 \cdot \frac{2 \cdot \bar{v} - (1-q) \cdot (\sqrt{\bar{v}} - 1)}{\sqrt{\bar{v}} \cdot (1 + q \cdot (\sqrt{\bar{v}} - 1))} \\ e_3(v_3 = \bar{v}) &= C_1 \cdot \frac{2 \cdot \bar{v} + q \cdot \sqrt{\bar{v}} \cdot (\sqrt{\bar{v}} - 1)}{1 + q \cdot (\sqrt{\bar{v}} - 1)} \end{aligned}$$

$$u_1, u_2 \in \left[ C_1 \cdot \frac{1 + q \cdot (\bar{v} - 1)}{2 \cdot \bar{v}}, C_1 \cdot \frac{1 + 2 \cdot \bar{v} + q \cdot (\bar{v} - 1)}{2 \cdot \bar{v}} \right]$$

$$\max_{i \in \{1,2\}} \min u_i(a, v_i) = C_1 \cdot \frac{1 + q \cdot (\bar{v} - 1) + \bar{v}}{2 \cdot \bar{v}}, \; i = 1, 2 \; ,$$

## A.2 $(v_1, v_2) = (1, \bar{v})$ , $\mathcal{A} = A$

The exact solutions to conditions (1) and (2) are given by

$$\begin{aligned} e_1 &= 0 \\ e_2 &= \begin{cases} C_2 & \text{if } \bar{v} < \left(\frac{3-q}{1-q}\right)^2 \\ \bar{v} \cdot \left(\frac{1-q}{3-q}\right)^2 & \text{if } \bar{v} \geq \left(\frac{3-q}{1-q}\right)^2 \end{cases} \\ e_3(v_3 = 1) &= \begin{cases} C_2 \cdot \frac{2-(1-q)\cdot(\sqrt{\bar{v}}-1)}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \bar{v} < \left(\frac{3-q}{1-q}\right)^2 \\ 0 & \text{if } \bar{v} \geq \left(\frac{3-q}{1-q}\right)^2 \end{cases} \\ e_3(v_3 = \bar{v}) &= \begin{cases} C_2 \cdot \frac{2+q\cdot\sqrt{\bar{v}}\cdot(\sqrt{\bar{v}}-1)}{1+q\cdot(\sqrt{\bar{v}}-1)} & \text{if } \bar{v} < \left(\frac{3-q}{1-q}\right)^2 \\ \bar{v} \cdot \frac{1-q}{(3-q)^2} \cdot 2 & \text{if } \bar{v} \geq \left(\frac{3-q}{1-q}\right)^2 \end{cases} \\ u_1 &= \begin{cases} C_2 \cdot \frac{3+q\cdot(\bar{v}-1)}{2\cdot\bar{v}} & \text{if } \bar{v} < \left(\frac{3-q}{1-q}\right)^2 \\ \frac{1}{2} \cdot \frac{1+q}{3-q} & \text{if } \bar{v} \geq \left(\frac{3-q}{1-q}\right)^2 \end{cases} \\ u_2 &= \begin{cases} C_2 \cdot \frac{1+q\cdot(\bar{v}-1)}{2} & \text{if } \bar{v} < \left(\frac{3-q}{1-q}\right)^2 \\ \frac{\bar{v}}{2} \cdot \frac{1+6\cdot q-3\cdot q^2}{(3-q)^2} & \text{if } \bar{v} \geq \left(\frac{3-q}{1-q}\right)^2 \end{cases} \end{aligned}$$

## A.3 $(v_1, v_2) = (1, \bar{v})$, $\mathcal{A} = \neg A$

The exact solutions to condition (1) are given by

$$e_1 = \begin{cases} C_3 \cdot \frac{2-\bar{v}+q\cdot(\bar{v}-1)}{1+q\cdot(\bar{v}-1)} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ 0 & \text{if } \frac{2-q}{1-q} \leq \bar{v} \end{cases}$$

$$e_2 = \begin{cases} C_3 \cdot \frac{\bar{v}+q\cdot(\bar{v}-1)}{1+q\cdot(\bar{v}-1)} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ \tilde{C}_3 & \text{if } \frac{2-q}{1-q} \leq \bar{v} < \left(\frac{2-q}{1-q}\right)^2 \\ \left(\frac{1-q}{2-q}\right)^2 \cdot \bar{v} & \text{if } \left(\frac{2-q}{1-q}\right)^2 \leq \bar{v} \end{cases}$$

$$e_3(v_3 = 1) = \begin{cases} C_3 \cdot \frac{1+(1-q)\cdot(\sqrt{\bar{v}}-1)^2}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ \tilde{C}_3 \cdot \frac{2-\sqrt{\bar{v}}+q\cdot(\sqrt{\bar{v}}-1)}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \frac{2-q}{1-q} \leq \bar{v} < \left(\frac{2-q}{1-q}\right)^2 \\ 0 & \text{if } \left(\frac{2-q}{1-q}\right)^2 \leq \bar{v} \end{cases}$$

$$e_3(v_3 = \bar{v}) = \begin{cases} C_3 \cdot \frac{\bar{v}+q\cdot(\sqrt{\bar{v}}-1)^2}{1+q\cdot(\sqrt{\bar{v}}-1)} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ \tilde{C}_3 \cdot \frac{1+q\cdot\sqrt{\bar{v}}\cdot(\sqrt{\bar{v}}-1)}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \frac{2-q}{1-q} \leq \bar{v} < \left(\frac{2-q}{1-q}\right)^2 \\ \frac{1-q}{(2-q)^2} \cdot \bar{v} & \text{if } \left(\frac{2-q}{1-q}\right)^2 \leq \bar{v} \end{cases}$$

$$u_1 = \begin{cases} C_3 \cdot \frac{(2-\bar{v}+q\cdot(\bar{v}-1))^2}{2\cdot\bar{v}\cdot(1+q\cdot(\bar{v}-1))} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ 0 & \text{if } \bar{v} \geq \frac{2-q}{1-q} \end{cases}$$

$$u_2 = \begin{cases} C_3 \cdot \frac{(\bar{v}+q\cdot(\bar{v}-1))^2}{2\cdot(1+q\cdot(\bar{v}-1))} & \text{if } \bar{v} < \frac{2-q}{1-q} \\ \tilde{C}_3 \cdot \frac{2-\bar{v}+q\cdot(\bar{v}-1)}{\bar{v}} & \text{if } \frac{2-q}{1-q} \leq \bar{v} < \left(\frac{2-q}{1-q}\right)^2 \\ q \cdot \bar{v} + (1-q) \cdot \left(\frac{1-q}{2-q}\right)^2 \cdot \bar{v} & \text{if } \left(\frac{2-q}{1-q}\right)^2 \leq \bar{v} \end{cases} .$$

## A.4 $(v_1, v_2) = (\bar{v}, 1)$, $\mathcal{A} = A$

This case is symmetric to $(v_1, v_2) = (1, \bar{v})$ , $\mathcal{A} = A$ in appendix A.2.

## A.5 $(v_1, v_2) = (\bar{v}, 1)$, $\mathcal{A} = \neg A$

This case is symmetric to $(v_1, v_2) = (1, \bar{v})$, $\mathcal{A} = \neg A$ in appendix A.3.

## A.6 $(v_1, v_2) = (\bar{v}, \bar{v})$, $\mathcal{A} = \neg A$

The exact solutions to condition (1) are given by

$$e_1 = e_2 = \begin{cases} 2 \cdot C_2 & \text{if } \bar{v} < \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \\ \frac{1+\sqrt{(1+q)^3}-3\cdot q}{(3-q)^2} \cdot \bar{v} \cdot \frac{1-q}{1+q} + \frac{q}{1+q} \cdot \frac{\bar{v}}{2} & \text{if } \bar{v} \geq \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \end{cases}$$

$$e_3(v_3 = 1) = \begin{cases} 2 \cdot C_2 \cdot \frac{\bar{v}-(1+q)\cdot(\sqrt{\bar{v}}-1)^2}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \bar{v} < \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \\ 0 & \text{if } \bar{v} \geq \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \end{cases}$$

$$e_3(v_3 = \bar{v}) = \begin{cases} 2 \cdot C_2 \cdot \frac{1+q\cdot(\sqrt{\bar{v}}-1)^2}{1+q\cdot(\sqrt{\bar{v}}-1)} & \text{if } \bar{v} < \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \\ & \text{if } \bar{v} \geq \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \end{cases}$$

$$u_1 = u_2 = \begin{cases} C_2 \cdot (1 + q \cdot (\bar{v} - 1)) & \text{if } \bar{v} < \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \\ \frac{1+\sqrt{(1+q)^3}-3\cdot q}{(3-q)^2} \cdot \bar{v} & \text{if } \bar{v} \geq \left(\frac{q+\sqrt{1+q}-1}{q}\right)^2 \end{cases} .$$

## A.7 $(v_1, v_2) = (\bar{v}, \bar{v})$, $\mathcal{A} = A$

Conditions (1) and (2) are satisfied by all $e_1, e_2 \geq 0$ such that

$$e_1 + e_2 = \begin{cases} C_2 & \text{if } \bar{v} \leq \left(\frac{3-q}{1-q}\right)^2 \\ \bar{v} \cdot \left(\frac{1-q}{3-q}\right)^2 & \text{if } \bar{v} > \left(\frac{3-q}{1-q}\right)^2 \end{cases}$$

$$e_3(v_3 = 1) = \begin{cases} C_2 \cdot \frac{2-(1-q)\cdot(\sqrt{\bar{v}}-1)}{\sqrt{\bar{v}}\cdot(1+q\cdot(\sqrt{\bar{v}}-1))} & \text{if } \bar{v} \leq \left(\frac{3-q}{1-q}\right)^2 \\ 0 & \text{if } \bar{v} > \left(\frac{3-q}{1-q}\right)^2 \end{cases}$$

$$e_3(v_3 = \bar{v}) = \begin{cases} C_2 \cdot \frac{2+q\cdot(\sqrt{\bar{v}}-1)\cdot\sqrt{\bar{v}}}{1+q\cdot(\sqrt{\bar{v}}-1)} & \text{if } \bar{v} \leq \left(\frac{3-q}{1-q}\right)^2 \\ \bar{v} \cdot \left(\frac{1-q}{3-q}\right)^2 \cdot \frac{2}{1-q} & \text{if } \bar{v} > \left(\frac{3-q}{1-q}\right)^2 \end{cases}$$

$$u_1, u_2 \in \begin{cases} \left[C_2 \cdot \frac{1+q\cdot(\bar{v}-1)}{2}, C_2 \cdot \frac{3+q\cdot(\bar{v}-1)}{2}\right] & \text{if } \bar{v} \leq \left(\frac{3-q}{1-q}\right)^2 \\ \left[\frac{\bar{v}}{2} \cdot \frac{1+6q-3q^2}{(3-q)^2}, \frac{\bar{v}}{2} \cdot \frac{3+2q-q^2}{(3-q)^2}\right] & \text{if } \bar{v} > \left(\frac{3-q}{1-q}\right)^2 \end{cases}$$

and

$$\min\ \max\{u_1, u_2\} = \begin{cases} C_2 \cdot \frac{2+q\cdot(\bar{v}-1)}{2} & \text{if } \bar{v} \leq \left(\frac{3-q}{1-q}\right)^2 \\ \bar{v} \cdot \frac{1+2q-q^2}{(3-q)^2} & \text{if } \bar{v} > \left(\frac{3-q}{1-q}\right)^2 \end{cases} .$$

In this case, the optimal choices of players one and two are not determined uniquely.

# B Only The Strong Stand Alone – Optimal Efforts After A Unilateral Deviation In Stage 1

We derive the optimal effort choices that follow a unilateral deviation from $\hat{a}_1$ to $\tilde{a}_1$ in stage 1 numerically and therefore need to assign specific numbers to the parameters. We choose $\bar{v} = 2$ and $q = \frac{1}{2}$. Players two and three believe that player one uses $\hat{a}_1(v_1 = 1) = yes$ and $\hat{a}_1(v_1 = 2) = no$. Table 4 lists the function values of the optimal effort choices of players two and three if they believe to be in one of the six cases for the parameter values $\bar{v} = 2$ and $q = \frac{1}{2}$. The corresponding functions are presented in appendix A.

| case | $(v_1, v_2)$ | $(a_1, a_2, \mathcal{A})$ | $e_2(a, v_2)$ | $e_3(a, v_3 = 1)$ | $e_3(a, v_3 = 2)$ |
|---|---|---|---|---|---|
| #1: | $(1, 1)$ | $(yes, yes, A)$ | 0.0482 | 0.2140 | 0.3426 |
| #2: | $(1, 2)$ | $(yes, no, A)$ | 0.2379 | 0.2498 | 0.4519 |
| #3: | $(1, 2)$ | $(yes, no, \neg A)$ | 0.4797 | 0.1831 | 0.4973 |
| #4: | $(2, 1)$ | $(no, yes, A)$ | 0 | 0.2498 | 0.4519 |
| #5: | $(2, 1)$ | $(no, yes, \neg A)$ | 0.0959 | 0.1831 | 0.4973 |
| #6: | $(2, 2)$ | $(no, no, \neg A)$ | 0.4758 | 0.0239 | 0.4280 |

Table 4: optimal effort choices of players two and three who believe that $v_1 = 1$, if $a_1 = yes$ and $v_1 = \bar{v}$, if $a_1 = no$ for $\bar{v} = 2$ and $q = \frac{1}{2}$.

Figure 9 illustrates the best responses of player one to the choices of players two and three who believe to be in the cases one to six.

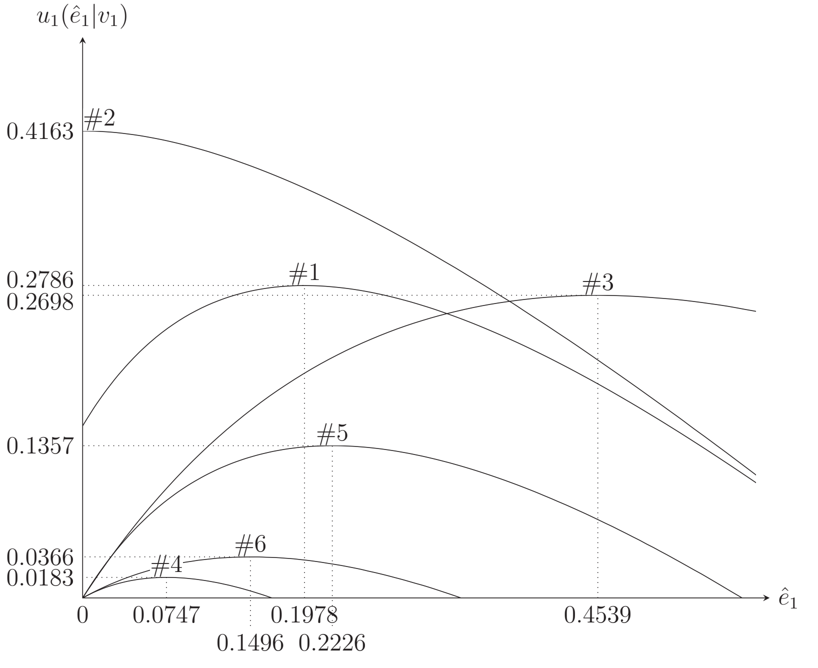


Figure 9: best responses of player one to the choices listed in table 4

# C Only The Weak Stand Alone – Optimal Effort Choices After A Unilateral Deviation In Stage 1

We derive the optimal effort choices that follow a unilateral deviation from $\hat{a}_1$ to $\tilde{a}_1$ in stage 1 numerically and therefore need to assign specific numbers to the parameters. We choose $\bar{v} = 2$ and $q = \frac{1}{2}$. Players two and three believe that player one uses $\hat{a}_1(v_1 = 1) = yes$ and $\hat{a}_1(v_1 = 2) = no$. Table 4 lists the function values of the optimal effort choices of players two and three if they believe to be in one of the six cases for the parameter values $\bar{v} = 2$ and $q = \frac{1}{2}$. The corresponding functions are presented in appendix A.

| case | $(v_1, v_2)$ | $(a_1, a_2, \mathcal{A})$ | $e_2(a, v_2)$ | $e_3(a, v_3 = 1)$ | $e_3(a, v_3 = 2)$ |
|---|---|---|---|---|---|
| #0: | $(1, 1)$ | $(no, no, \neg A)$ | 0.1927 | 0.2354 | 0.4925 |
| #2: | $(1, 2)$ | $(no, yes, A)$ | 0.2379 | 0.2498 | 0.4519 |
| #3: | $(1, 2)$ | $(no, yes, \neg A)$ | 0.4797 | 0.1831 | 0.4973 |
| #4: | $(2, 1)$ | $(yes, no, A)$ | 0 | 0.2498 | 0.4519 |
| #5: | $(2, 1)$ | $(yes, no, \neg A)$ | 0.0959 | 0.1831 | 0.4973 |
| #7: | $(2, 2)$ | $(yes, yes, A)$ | 0.1189 | 0.2498 | 0.4519 |

Table 5: optimal effort choices of players two and three who believe that $v_1 = \bar{v}$, if $a_1 = yes$ and $v_1 = 1$, if $a_1 = no$ for $\bar{v} = 2$ and $q = \frac{1}{2}$.

Figure 10 illustrates the best responses of player one to the choices of players two and three who believe to be in the cases #0 to #7.

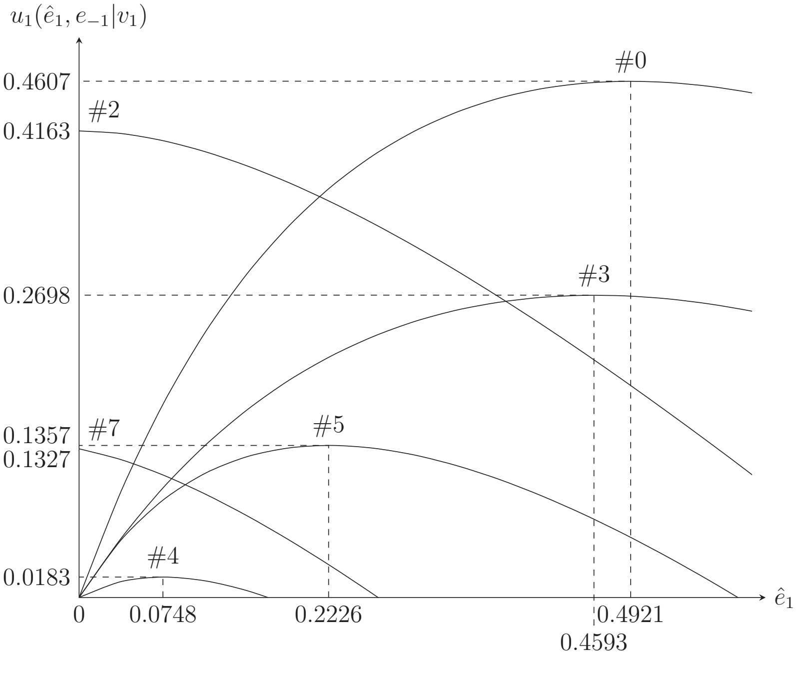


Figure 10: best responses of player one to the choices listed in table 5

# D Proofs of section 3.4

**Lemma 2** *If $e(1)$ and $e(\bar{v})$ solve the inequalities (3) and (4), then $e(1)$ and $e(\bar{v})$ are positive and the inequalities are binding.*

PROOF : The left hand side of (3) tends to $\infty$ for $e(1) \to 0$ and the left hand side of (4) tends to $\infty$ for $e(\bar{v}) \to 0$. Hence $e(1), e(\bar{v}) > 0$ and both inequalities must be binding. □

The existence of an equilibrium heavily relies on the existence of the intersection of the two curves defined by (3) and (4). The next lemma provides an implicit definition of these curves.

**Lemma 3** *Conditions (3) and (4) imply a unique well defined continuously differentiable function $g : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ such that $e(1) = g_1(\cdot, e_3(1))$ and $e_3(1) = g_2(e(1), \cdot)$.*

PROOF : Use (3) and (4) and lemma 2 to define the continuous function $f : \mathbb{R}^2_+ \times \mathbb{R}^2_{++} \to \mathbb{R}^2$ as

$$f(x_1, x_2, y_1, y_2) = \begin{pmatrix} q^2 \frac{1}{9y_1} + q(1-q)\frac{y_1+x_2}{(2y_1+x_2)^2} + (1-q)^2 \frac{x_2}{(y_1+2x_2)^2} - \frac{1}{2} \\ q^2 \frac{x_1}{(y_2+2x_1)^2} + q(1-q)\frac{x_1+y_2}{(x_1+2y_2)^2} + (1-q)^2 \frac{1}{9y_2} - \frac{1}{2\bar{v}} \end{pmatrix} .$$

For $(a, b) \in \mathbb{R}^2_+ \times \mathbb{R}^2_{++}$ we have that

$$\begin{aligned}
\frac{\partial f_1}{\partial x_2}(a,b) &= -q \cdot (1-q) \cdot \frac{a_2}{(2b_1+a_2)^3} - (1-q)^2 \cdot \frac{2a_2 - b_1}{(b_1+2a_2)^3} \\
\frac{\partial f_2}{\partial x_1}(a,b) &= q^2 \cdot \frac{b_2 - 2a_1}{(b_2+2a_1)^3} - q \cdot (1-q) \cdot \frac{a_1}{(a_1+2b_2)^3} \\
\frac{\partial f_1}{\partial y_1}(a,b) &= -q^2 \frac{1}{9b_1^2} - q(1-q)\frac{2b_1+3a_2}{(2b_1+a_2)^3} - (1-q)^2 \frac{a_2}{(b_1+2a_2)^3} \\
\frac{\partial f_2}{\partial y_2}(a,b) &= -q^2 \frac{a_1}{(b_2+2a_1)^3} - q(1-q)\frac{3a_1+2b_2}{(a_1+2b_2)^3} - (1-q)^2 \frac{1}{9b_2^2}
\end{aligned}$$

and

$$\frac{\partial f_1}{\partial x_1}(a,b) = \frac{\partial f_2}{\partial x_2}(a,b) = \frac{\partial f_1}{\partial y_2}(a,b) = \frac{\partial f_2}{\partial y_1}(a,b) = 0 .$$

Hence the Jacobian

$$\frac{\partial f}{\partial y}(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & 0 \\ 0 & \frac{\partial f_2}{\partial y_2} \end{pmatrix}(a,b)$$

has strictly negative entries on the diagonal. Note that $f(x,y) \underset{y_1,y_2\to\infty}{\longrightarrow} \left(-\frac{1}{2}, -\frac{1}{2\cdot\bar{v}}\right)$ $\forall\ x \in \mathbb{R}^2_+$. As $f_i(x,y) \underset{y_i\to 0}{\longrightarrow} \infty$, $i = 1,2$ and by continuity of $f(\cdot,\cdot)$, we have that for each $a \in \mathbb{R}^2_+$ there exists a unique $b \in \mathbb{R}^2_{++}$ such that $f(a,b) = 0$. Because the Jacobian of $f$ is invertible for all $(a,b) \in \mathbb{R}^2_+ \times \mathbb{R}^2_{++}$, the implicit function theorem[9] implies that there exists a unique continuously differentiable function $g : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that

$$\left\{(x, g(x)) | x \in \mathbb{R}^2_+\right\} = \left\{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_{++} | f(x,y) = 0\right\}$$

with

$$\frac{\partial g(x)}{\partial x} = -\begin{pmatrix} 0 & \frac{\partial f_1}{\partial x_2} \Big/ \frac{\partial f_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_1} \Big/ \frac{\partial f_2}{\partial y_2} & 0 \end{pmatrix} (x, g(x)) \ .$$

□

**Lemma 4 (Proposition 3)** *For each $\bar{v} \geq 1$ and $q \in (0,1)$ there exists a pair $(e^*(1), e^*(\bar{v}))$ such that conditions (3) and (4) are satisfied.*

Proof : For $g : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ as defined in the proof of lemma 3 we have $g(0) = (\frac{2}{9}q^2 + \frac{1}{2}q(1-q), \frac{2}{9}(1-q)^2\bar{v} + \frac{1}{2}q(1-q)\bar{v})$ and $g(x) \underset{x\to\infty}{\longrightarrow} \frac{2}{9}(q^2, \bar{v}(1-q)^2)$. By continuity, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded by some $(K,K) \in \mathbb{R}^2$. By Brouwer's fixed-point theorem there exists a point $x^* \in [0,K]^2$ with $g(x^*) = x^*$ for $K < \infty$ large enough. Define $e^*(1) = x^*_1$ and $e^*(\bar{v}) = x^*_2$. Note that lemmata 2, 3 or 4 are valid for any $q \in (0,1)$ and $\bar{v} \geq 1$. □

**Lemma 5** *If $e^*(1)$ and $e^*(\bar{v})$ satisfy conditions (3) and (4), then $e^*(\bar{v}) > e^*(1)$ for all $\bar{v} > 1$.*

Proof : Consider some fixed point $x$ of $g(\cdot)$ with $x_1 \geq x_2$.

$$\begin{aligned} \frac{x_1}{(x_1 + 2\cdot x_1)^2} &\leq \frac{x_1}{(x_2 + 2x_1)^2} \\ \frac{x_1 + x_2}{(x_1 + x_1 + x_2)^2} &\leq \frac{x_1 + x_2}{(x_2 + x_1 + x_2)^2} \\ \frac{x_2}{(x_1 + 2x_2)^2} &\leq \frac{x_2}{(x_2 + 2x_2)^2} \ . \end{aligned}$$

[9] See Munkres (1994).

As $-\frac{1}{2} < -\frac{1}{2\bar{v}}\ \forall\ \bar{v} > 1$, we have that $f_1(x, g(x)) < f_2(x, g(x))$, a contradiction to the definition of $g(\cdot)$. Hence $x_1 < x_2$. □

**Theorem 1** *For each* $\bar{v} \geq 1$ *and* $q \in (0, 1)$ *there exists a unique pair* $(e^*(1), e^*(\bar{v}))$ *such that conditions* (3) *and* (4) *are satisfied.*

Proof : We show that at any fixed point $x^*$ and in the direction of increasing $x_2$, the graph of $g_1(\cdot)$ intersects the graph of $g_2(\cdot)$ coming from $\{(x_1, x_2) : x_2 < g_2(x)\}$ which is the shaded area in figure 11.

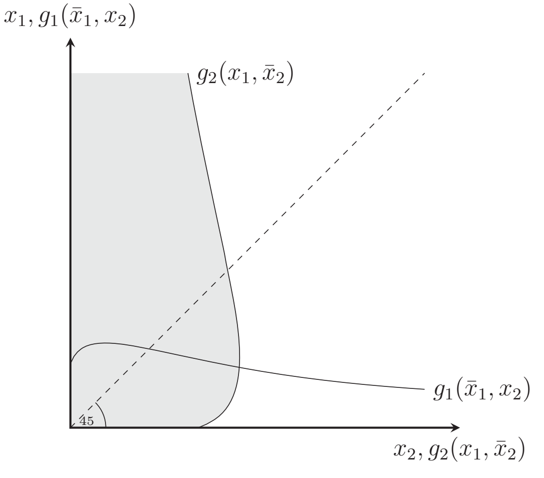


Figure 11: At any fixed point the graph of $g_1(\cdot)$ must hit the graph of $g_2(\cdot)$ from the direction of the shaded area.

If $\frac{\partial g_2}{\partial x_1}(x, g(x)) = 0$, this is trivially the case as $g_1(\cdot)$ has finite first derivatives at any point $x$. If $\frac{\partial g_2}{\partial x_1}(x^*, g(x^*)) > 0$, we need to show that $\frac{\partial g_1}{\partial x_2}(x^*, g(x^*))$ is strictly lower than the inverse of the slope of $g_2(\cdot)$ at $x^*$ and if $\frac{\partial g_2}{\partial x_1}(x^*, g(x^*)) < 0$ we need to show that the slope of $\frac{\partial g_1}{\partial x_2}(x^*, g(x^*))$ is strictly greater than the inverse of the slope of $g_2(\cdot)$ at $x^*$. This amounts to the condition

$$\frac{\partial g_1}{\partial x_2}(x^*, g(x^*)) \cdot \frac{\partial g_2}{\partial x_1}(x^*, g(x^*)) < 1 \tag{9}$$

at any fixed point $x^*$. If there were multiple fixed points, the graph of $g_1(\cdot)$ would have to interset $g_2(\cdot)$ coming from the white area for at least one fixed point, a contradiction. Lemma 3 states the partial derivatives of $g(\cdot)$ and by lemma 5 we know that any fixed point satisfies $x_2^* > x_1^*$ and hence $\frac{\partial f_1}{\partial x_2}(x^*, g(x^*)) < 0$ and $\frac{\partial f_2}{\partial y_2}(x^*, g(x^*)) < 0$. With these statements we can show that condition (9) is equivalent to

$$det\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = -\frac{\partial f_2}{\partial x_1} \cdot \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} > 0$$

for any fixed point $x^*$, where we use the g-determinant for $2 \times n$ matrices as defined by Radić (2005). Elementary rearrangements imply

$$-\frac{\partial f_2}{\partial x_1} \cdot \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} =$$

$$\frac{q^3(1-q)}{(x_2+2x_1)^6} \cdot \left(x_2^2 + 2x_1^2 + x_1 x_2\right) + \frac{q(1-q)^3}{(x_1+2x_2)^6}\left(x_1 x_2 + 2x_2^2 + x_1^2\right)$$
$$+8\frac{q^2(1-q)^2}{(x_2+2x_1)^3(x_1+2x_2)^3} \cdot \left(x_2^2 + x_1^2 + x_1 x_2\right)$$
$$+q^4 \frac{1}{9x_1^2}\frac{x_1}{(x_2+2x_1)^3} + q^3(1-q)\frac{1}{9x_1^2}\frac{3x_1+2x_2}{(x_1+2x_2)^3} + q^2(1-q)^2\frac{1}{9x_1^2}\frac{1}{9x_2^2}$$
$$+q(1-q)^3 \cdot \frac{2x_1+3x_2}{(2x_1+x_2)^3}\frac{1}{9x_2^2} + (1-q)^4\frac{x_2}{(x_1+2x_2)^3}\frac{1}{9x_2^2} > 0$$

□

# E Proof of section 3.5

Proof of Proposition 5:
Use the binding conditions (5) to (8) to define the continuous function $F_\alpha : \mathbb{R}^4 \times \mathbb{R}_+^4 \to \mathbb{R}^4$ as[10]

$$F_\alpha(u_1, u_2, u_3, u_4, y_1, y_2, y_3, y_4) =$$

[10]Nti (1997) and Franke and Ozturk (2009) use a positive constant in the denominator to handle discontinuity problems.

$$\begin{pmatrix} q^2\cdot\frac{y_3+\alpha}{(2\cdot u_1+y_3)^2+\alpha^2}+q\cdot(1-q)\cdot\frac{y_4+\alpha}{(2\cdot u_1+y_4)^2+\alpha^2}+(1-q)\cdot q\cdot\frac{y_3+\alpha}{(u_1+y_2+y_3)^2+\alpha^2}+(1-q)^2\cdot\frac{y_4+\alpha}{(u_1+y_2+y_4)^2+\alpha^2}-2 \\ q^2\cdot\frac{y_3+\alpha}{(u_2+y_1+y_3)^2+\alpha^2}+q\cdot(1-q)\cdot\frac{y_4+\alpha}{(u_2+y_1+y_4)^2+\alpha^2}+(1-q)\cdot q\cdot\frac{y_3+\alpha}{(2\cdot u_2+y_3)^2+\alpha^2}+(1-q)^2\cdot\frac{y_4+\alpha}{(2\cdot u_2+y_4)^2+\alpha^2}-\frac{2}{\bar{v}} \\ q^2\cdot\frac{2y_1+\alpha}{(u_3+2\cdot y_1)^2+\alpha^2}+q\cdot(1-q)\cdot\frac{2(y_1+y_2)+\alpha}{(u_3+y_1+y_2)^2+\alpha^2}+(1-q)^2\cdot\frac{2y_2+\alpha}{(u_3+2\cdot y_2)^2+\alpha^2}-1 \\ q^2\cdot\frac{2y_1+\alpha}{(u_4+2\cdot y_1)^2+\alpha^2}+q\cdot(1-q)\cdot\frac{2(y_1+y_2)+\alpha}{(u_4+y_1+y_2)^2+\alpha^2}+(1-q)^2\cdot\frac{2y_2+\alpha}{(u_4+2\cdot y_2)^2+\alpha^2}-\frac{1}{\bar{v}} \end{pmatrix}$$

and $G_\alpha : \mathbb{R}^4 \times \mathbb{R}^4_+ \to \mathbb{R}^4$ with

$$G_\alpha(u, y) = F_\alpha(u^+, y) + u^+ - u$$

for $u \in \mathbb{R}^4$ and $y \in \mathbb{R}^4_+$.

Consider any $\hat{y} \in \mathbb{R}^4$.
Suppose $G_{\alpha j}(0, \hat{y}) > 0$. As $\lim_{u_j \to \infty} G_{\alpha j}(u, \hat{y}) \in (0, -\infty)$ for each $j$ and as $G_{\alpha j}(u, \hat{y})$ is continuous in $u_j$, there exists some $\hat{u}_j$ such that $G_{\alpha j}(\hat{u}, \hat{y}) = 0$.
Suppose $G_{\alpha j}(0, \hat{y}) < 0$. Then $G_{\alpha j}(0, \hat{y}) = F_{\alpha j}(0, \hat{y}) < 0$. Define $\hat{u}$ such that $\hat{u}_j = F_{\alpha j}(0, \hat{y})$. Then $G_{\alpha j}(\hat{u}, \hat{y}) = F_{\alpha j}(0, \hat{y}) - F_{\alpha j}(0, \hat{y}) = 0$.
Therefore, for each $\hat{y} \in \mathbb{R}^4$ there exist a $\hat{u}_\alpha \in \mathbb{R}^4$ such that $G_\alpha(\hat{u}_\alpha, \hat{y}) = 0$.

We consider now the matrix $\frac{\partial^+ G_\alpha}{\partial u}(u, y) \in \mathbb{R}^4 \times \mathbb{R}^4$ with

$$\frac{\partial^+ G_{\alpha i}}{\partial u_j} = \lim_{h \searrow 0} \frac{G_{\alpha i}(u + h \cdot e_j, y) - G_{\alpha i}(u, y)}{h}, i, j = 1, \ldots, 4 \ .$$

For $u \in \mathbb{R}^4$ with $u_j \geq 0$ we have

$$\frac{\partial^+ G_{\alpha i}}{\partial u_j} = \begin{cases} \frac{\partial F_{\alpha i}}{\partial u_i} & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} .$$

For $u \in \mathbb{R}^4$ with $u_j < 0$ we have

$$\frac{\partial^+ G_{\alpha i}}{\partial u_j} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} .$$

Note that in our special setup we can simplify the general analysis of Robinson (1991).

As $\frac{\partial F_{\alpha i}}{\partial u_i}(u, y) < 0$ for all $(u, y) \in \mathbb{R}^4 \times \mathbb{R}^4_+$, the matrix $\frac{\partial^+ G_\alpha}{\partial u}(u, y)$ has full rank for all $(u, y) \in \mathbb{R}^4 \times \mathbb{R}^4_+$. By the implicit function theorem, for any point $(\hat{u}, \hat{y}) \in \mathbb{R}^4 \times \mathbb{R}^4_+$ with $G_\alpha(\hat{u}, \hat{y}) = 0$ there is a neighborhood $H$ of $y_0$, $H \subset \mathbb{R}^4_+$ and a unique continuous function $g_\alpha : H \to \mathbb{R}^4$ such that $g_\alpha(\hat{y}) = \hat{u}$ and $G_\alpha(g_\alpha(y), y) = 0 \ \forall \ y \in H$.

As $\frac{\partial F_{\alpha i}}{\partial y_j}(u,y) \leq 0$ for all $i,j = 1,\ldots,4$ there exists a finite $K \in \mathbb{R}_+$ such that $g_\alpha(y) \in [0,K]^4$ for all $y \in \mathbb{R}^4_+$ and by Brouwer's fixed point theorem there exists some $y^*_\alpha \in [0,K]^4$ such that $y^*_\alpha = g_\alpha(y^*_\alpha)$.

For each fixed point $y^*_\alpha$ we have that $F_{\alpha i}(g_\alpha(y^*_\alpha)^+, y^*_\alpha) \leq 0$ for $i = 1,\ldots,4$ and by lemma 1 $F_{\alpha i}(g_\alpha(y^*_\alpha)^+, y^*_\alpha) < 0$ implies $g_\alpha(y^*_\alpha)^+_i = 0$. Therefore $g_\alpha(y^*_\alpha)^+$ satisfies statement $i)$ of the lemma for $\alpha > 0$.

For the following arguments consider some $\alpha > 0$ and suppose there exists some $y^*_\alpha \in \mathbb{R}^4_+$ with $F_\alpha(y^*_\alpha, y^*_\alpha) \leq 0$ and $F_{\alpha i}(y^*_\alpha, y^*_\alpha) < 0 \Rightarrow y^*_{\alpha i} = 0$.

Suppose $y^*_{\alpha 1} = y^*_{\alpha 3} = 0$. Then there exists some $\underline{\alpha} > 0$ such that $F_{\alpha 1}(g_\alpha(y^*_\alpha), y^*_\alpha) > 0 \ \forall \ 0 < \alpha < \underline{\alpha}$, a contradiction. Hence $y^*_{\alpha 1} + y^*_{\alpha 3} > 0$.

Suppose $y^*_{\alpha 1} > y^*_{\alpha 2} \geq 0$. Then $F_{\alpha 1}(g_\alpha(y^*_\alpha), y^*_\alpha) < F_{\alpha 2}(g_\alpha(y^*_\alpha), y^*_\alpha) \leq 0$, a contradiction to $y^*_{\alpha 1} > 0$. Hence $y^*_{\alpha 1} \leq y^*_{\alpha 2}$.

Suppose $y^*_{\alpha 3} > y^*_{\alpha 4} \geq 0$. Then $F_{\alpha 3}(g_\alpha(y^*_\alpha), y^*_\alpha) < F_{\alpha 4}(g_\alpha(y^*_\alpha), y^*_\alpha) \leq 0$, a contradiction to $y^*_{\alpha 3} > 0$. Hence $y^*_{\alpha 3} \leq y^*_{\alpha 4}$.

Suppose $y^*_{\alpha 2} = 0$. Then $y^*_{\alpha 1} = 0$ and $F_{\alpha 3}(g_\alpha(y^*_\alpha), y^*_\alpha) < 0$ and hence $y^*_{\alpha 3} = 0$, a contradiction to $y^*_{\alpha 1} + y^*_{\alpha 3} > 0$. Hence $y^*_{\alpha 2} > 0$.

Suppose $y^*_{\alpha 4} = 0$. Then $y^*_{\alpha 3} = 0$ and $F_{\alpha 1}(g_\alpha(y^*_\alpha), y^*_\alpha) < 0$ and hence $y^*_{\alpha 1} = 0$, a contradiction to $y^*_{\alpha 1} + y^*_{\alpha 3} > 0$. Hence $y^*_{\alpha 4} > 0$.

Therefore $y^*_{\alpha i} + y^*_{\alpha j} > 0$ for any $i,j = 1,\ldots,4$, $i \neq j$. As $F_\alpha(u,y)$ is continuous in $\alpha$ at any $(u,y) = (y^*_\alpha, y^*_\alpha)$ with $y^*_{\alpha i} + y^*_{\alpha j} > 0$, $i,j = 1,\ldots,4$, $i \neq j$, we have that $y^*_\alpha \underset{\alpha \to 0}{\to} e^* = (e^*(1), e^*_3(1), e^*(\bar{v}), e^*_3(\bar{v}))$, where $e^*$ satisfies statements i) and ii) of the lemma.

Suppose $\lim\limits_{\bar{v}\to\infty} e^*(\bar{v}) < \infty$. As $\frac{8}{\bar{v}} \underset{\bar{v}\to\infty}{\longrightarrow} 0$ and as $e^*(1) < e^*(\bar{v})$ we have that $e^*_3(1) \underset{\bar{v}\to\infty}{\longrightarrow} \infty$ such that (6) can be satisfied. But then the inequality in (7) is strict and $e^*_3(1) = 0$, a contradiction. Analogue arguments imply that $e^*_3(\bar{v}) \underset{\bar{v}\to\infty}{\longrightarrow} \infty$. Therefore we have

$$(5) \quad \underset{\bar{v}\to\infty}{\longrightarrow} \quad q^2 \cdot \frac{e_3(1)}{(2e(1) + e_3(1))^2} \leq 2$$

$$(7) \quad \underset{\bar{v}\to\infty}{\longrightarrow} \quad q^2 \cdot \frac{2e(1)}{(2e(1) + e_3(1))^2} \leq 1$$

Observe that (5) is strict implies $e(1) = 0$ which implies (7) is strict which implies $e_3(1) = 0$ which implies that (7) is strict, hence $e(1) \underset{\bar{v}\to\infty}{\longrightarrow} 0 \Leftrightarrow e_3(1) \underset{\bar{v}\to\infty}{\longrightarrow} 0$. Hence, if $e(1)$ or $e_3(1) \underset{\bar{v}\to\infty}{\longrightarrow} 0$ then $e(1) + e_3(1) \underset{\bar{v}\to\infty}{\longrightarrow} 0$, a contradiction. Hence $\lim\limits_{\bar{v}\to\infty} e(1), e_3(1) > 0$ and the inequalities in (5) and (7) must be binding for $\bar{v} \to \infty$. The unique solution to these two equations is $e^*(1) = q^2 \cdot \frac{1}{18}$ and $e_3^*(1) = q^2 \cdot \frac{2}{9}$. This proves statement *iii* of the lemma.

$\lim_{\bar{v}\to 1} e(\bar{v}) = \lim_{\bar{v}\to 1} e(1)$ and $\lim_{\bar{v}\to 1} e_3(\bar{v}) = \lim_{\bar{v}\to 1} e_3(1)$ follows by symmetry. Conditions (6) collapses to (5) which reduces to $\frac{e_3(1)}{(2\cdot e(1)+e_3(1))^2} \leq 2$ and condition (8) collapses to (7) which reduces to $\frac{e(1)}{(2\cdot e(1)+e_3(1))^2} \leq \frac{1}{2}$. As before we have $\lim_{\bar{v}\to 1} e(\bar{v}), e_3(\bar{v}) > 0$ and both inequalities must be binding. The unique solution to these two equations is $e^*(1) = \frac{1}{18}$ and $e_3^*(1) = \frac{2}{9}$. This proves statement *iv* of the lemma. □

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