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A Non-Parametric Approach

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Jan Heufer and Per Hjertstrand[1]

Homothetic Efficiency

A Non-Parametric Approach

Abstract

This article provides a robust non-parametric approach to demand analysis based on a concept called homothetic efficiency. Homotheticity is a useful restriction or assumption but data rarely satisfy testable conditions. To overcome this problem, this article provides a way to estimate homothetic efficiency of consumption choices by consumers. The basic efficiency index suggested is similar to Afriat's (1972) efficiency index and Varian's (1993) violation index. It generalises Heufer's (2013b) two-dimensional concept to arbitrary dimensions and is motivated by a form of rationalisation similar to the one proposed by Halevy et al. (2012). The method allows to construct scalar factors which can be used to construct revealed preferred and worse sets. The approach also provides considerably more discriminatory power against irrational behaviour than standard utility maximisation. An application to experimental and household expenditure data illustrates how the method allows to recover preferences and increase test power.

JEL Classification: C14, D11, D12

Keywords: Demand theory; efficiency; experimental economics; homotheticity; non-parametric analysis; revealed preference; utility maximisation

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[1] Jan Heufer, TU Dortmund; Per Hjertstrand, Research Institute for Industrial Economics. - Views expressed represent exclusively the authors' own opinions. *Thanks to Wolfgang Leininger for his support. Access to data collected by Raymond Fisman, Shachar Kariv, and Daniel Markovits is gratefully acknowledged.* *– All correspondence to Jan Heufer, TU Dortmund, Department of Economics and Social Science, 44221 Dortmund, Germany, e-mail: jan.heufer@tu-dortmund.de*

1 INTRODUCTION

Homotheticity of consumer preferences is an important and useful concept in both theoretical and empirical work. If a consumer's preferences are homothetic, we can deduce his entire preference relation from a single indifference set. It therefore allows to recover much more of the preferences from a limited data set. Furthermore, testing data for homothetic utility maximisation can provide substantially stronger discriminatory power against alternative hypothesis than testing for utility maximisation alone. Homotheticity has important implications in many different fields of economics; for examples, aggregation of consumer demand and the existence of "community indifference curves", modelling of separable preference structures and its connection to two-stage budgeting, and as a common assumption in the international trade literature.

1.1 Summary of Contributions

The purpose of this article is to provide measures for the homothetic efficiency of a data set. We introduce the Homothetic Efficiency Index (Hei) which is a homothetic analogue to the well known Afriat Efficiency Index (Aei, also known as the Critical Cost Efficiency Index, Ccei) which can be interpreted as a measure of wasted income. The Hei generalises the index proposed by Heufer (2013b) for the two-dimensional case. We also extend this measure by introducing the Homothetic Efficiency Vector (Hev) which provides efficiency indices for each observed choice and allows for a more detailed and robust data analysis.

Varian's (1983) Homothetic Axiom of Revealed Preference (Harp) can be easily tested with a set of data. It is a necessary and sufficient condition for consistency with homothetic utility maximisation and therefore characterises the hypothesis of homothetic preferences. However, it is an unambigious test: Either the data satisfy Harp or not. When Harp is violated, the measures introduced here show how close the data come to being consistent with Harp. Both the Hei and the Hev provide the minimal adjustments which are required to make a data set consistent with homothetic utility maximisation. The measures are motivated by e- and $\mathbf{h}$-rationalisation which is similar to a concept recently introduced by Halevy et al. (2012). As Harp is a rather strong condition, it is often violated; without efficiency measures the only conclusion is then that the data is not perfectly consistent. Our measures allow us to go further by quantifying and interpreting the extent of the inconsistency.

We show how the Hei and the Hev can be used to recover more about a consumers' preference relation when a set of data comes reasonably close to homotheticity. This extends Varian's (1982) and Knoblauch's (1993) approach to non-parametric recoverability of preferences to a situation where data can be assumed to be the result of homothetic utility maximisation with minor errors.

To illustrate and motivate the methods put forward in this paper we apply them to two data sets. The first application is to data from an experimental dictator game conducted by Fisman et al. (2007). Using this data, we show how our methods can recover detailed information about subjects' preferences. The second application is to a panel of expenditures on non-durable consumption categories for 3,134 Spanish households. With this data, previously analysed in e.g. Browning and Collado (2001) and Crawford (2010) and Cherchye et al. (2014), we show that homothetic efficiency can be very high and still have considerably more discriminatory power against irrational behaviour than standard utility maximisation.

1.2 *Implications and Applications*

Testing and Test Power

Non-parametric tests for homotheticity in consumption and production theory have been considered in the literature before. Varian (1983) introduced HARP, an easily testable axiom that characterises monotonic and convex homothetic utility maximisation. Liu and Wong (2000) provided a stronger testable condition, which characterises strictly convex homothetic utility maximisation.

Testing data for consistency with Varian's (1982) Generalised Axiom of Revealed Preference (GARP), which is a necessary and sufficient condition for non-satiated utility maximisation, is as unambigious as the test for HARP. The data will either satisfy GARP or not. In cases where the data does not satisfy GARP, studies usually report an efficiency measure such as the AEI combined with various power measures. The most frequently used power measure was introduced by Bronars (1987) who suggested using Monte Carlo methods to compute the power of a test for GARP. One can generate many sets of random choices, usually from a uniform distribution on the budgets, and test these sets for consistency with GARP. The percentage of sets which do not satisfy GARP is the approximate test power.

One problem with the revealed preference approach is that the test power is sometimes very low. This is particuarly true when we wish to allow for small errors in decision making or for measurement error. For example, we might deem an AEI of 0.95 to be acceptable, but allowing for this extent of errors can lead to such a low power that the empirical analysis becomes almost meaningless. HARP is a stronger condition than GARP, and we can expect that it is far less likely that a set of random choices satisfies HARP. However, HARP will often also not be satisfied by the real data.[1] But homothetic efficiency can be very high for consumer choice data, as we demonstrate in the empirical part of the paper. Thus, homothetic efficiency may provide for an empirical analysis that has substantial discriminatory power against alternative hypothesis such as random behaviour, even when this is not the case for standard efficiency.

This is strongly supported by the results from our empirical applications. Specifically, our main results can be summarized as follows: (i) efficiency can be very high for HARP, thus providing motivation to assume homothetic preferences, (ii) HARP has noticably higher power than GARP for consumer choice data. For example, while the power of GARP can be below 10 percent for consumer choice data, the power of HARP is close to 100 percent, (iii) adjusting expenditure for efficiency in HARP has negligible effects on the power. Thus, HARP can have substantially higher power than GARP even when expenditure is adjusted for efficiency.

Heufer (2013a) provided a method to compute random choice data which satisfy GARP. This can be used to generate utility maximising choice sets that are tested to comply with HARP, which provides a conditional test power – the probability that a random choice set does not satisfy HARP given that is does satisfy GARP. We show that this is the case for both the experimental and consumer choice data.

Recoverability and Parametric Estimates

Varian (1982) described in detail the ways in which a researcher can recover everything that can be said about a consumer's preference based on a finite set of consumption data (see also Knoblauch 1992). Knoblauch (1993) extended Varian's approach to homothetic recoverability. Assuming homotheticity of preferences, if justified, allows the researcher to recover more information about the consumer's preferences.

Our approach allows for this extended recoverability even when homotheticity is violated by providing a way to adjust the data accordingly; with high homothetic efficiency, only minor adjustments are necessary. Our

[1] Manser and McDonald (1988) is a notable exception. They analyse U.S. consumption data from 1959 to 1985 on 101 commondities and find that homothetic preferences are consistent with this data.

empirical application demonstrate the usefulness of this approach by showing examples of revealed preferred and worse sets of subjects using data from the experiment carried out by Fisman et al. (2005). We show how a simple graphical analysis provides substantial information about the preferences of subjects.

Empirical work often focusses on estimating parameters of particular functional forms corresponding to homothetic utility functions, or demand systems (Cobb-Douglas and CES are two notable examples which are widely used in the literature). If the estimated utility function is homothetic, testing the data for consistency with HARP and computing efficiency measures can be employed as a robustness check or screening device. Consequently, it may be advisable not to estimate parameters in homothetic demand systems using data with low homothetic efficiency.

Recently, Halevy et al. (2012) introduced methods for parametric recoverability, where parameters are not estimated by minimising a statistical loss function but rather by maximising the money metric utility of a consumer, which is bounded by the revealed preference relation implicit in his set of choices. The extended homothetic recoverability we propose can lead to tighter bounds on the money metric utility function of a consumer and therefore allow for a better estimation of parameters, in particular those of homothetic utility functions.

Aggregation

Eisenberg (1961) showed that if income shares are fixed and every consumer's utility function is homothetic, then the market demand generated by individual utility maximisation is also generated by maximising a single homothetic utility function. This result shows that homotheticity of utility is a necessary condition for the existence of "community indifference curves" or "average preferences" (see also Gorman 1953). Thus, testing data for how close they come to homotheticity is easily motivated by the important implications for aggregation. See also Chipman (1965) for a discussion of demand aggregation in trade theory, Chipman (1974), Mantel (1976), Polemarchakis (1983), and Varian (1984b) on further issues of aggregation and homotheticity, and Shafer and Sonnenschein (1982) for a survey on market demand.

Two-Stage Budgeting, and Estimation of Demand Systems

It is standard practice when estimating large demand systems to impose separability restrictions. Often separable blocks are constructed in the form of a utility tree, usually referred to as the two-stage budgeting approach (Deaton and Muellbauer 1980). Gorman (1959) showed that, in general, homothetic separability (i.e., when sub-utility functions are homothetic) is a necessary and sufficient condition for two-stage budgeting. Thus, our methods can be used as pre-tests to check whether homothetic separability is a plausible assumption, and also provide guidance to which functional form be used for the sub-utility functions.

International Trade Theory

Theories of international trade typically assume that consumers have homothetic preferences, where the main purpose is to show that product differentiation, increasing returns and firm heterogeneity are decisive factors in explaining the extensive and intensive margins of international trade (see e.g. Krugman 1980, Melitz 2003, Helpman et al. 2008, Chaney 2008). The assumption of homothetic preferences in these models provide means and tools of analysing situations where technology rather than demand factors are the main driving force of aggregate outcomes. Assuming homotheticity also makes these models more tractable for empirical implementation. Thus, our methods can be used to test the underlying assumptions in the models.

Translation to Production Analysis

Hanoch and Rothschild (1972) and Varian (1984a) described non-parametric ways to test production for homotheticity.[2] Because this theory is very similar to testing for homotheticity in a consumer choice setting, our methods can be easily adapted to calculate homothetic efficiency of allocating factor inputs in production, and also recover detailed information about the underlying production technology. Varian (1984a) also considered tests for constant returns of scale in production analysis. Our methods can be adjusted accordingly to calculate homothetic efficiency for factor allocation in constant returns to scale technologies and recover information about such technologies. Finally, it is important to note that homothetic production functions, such as Cobb-Douglas and CES, are extensively used in, for example, empirical macroeconomics when modelling the economy's underlying production technology.[3]

1.3 Outline

The rest of the paper is organised as follows. Section 2 introduces the notation, recalls basic revealed preference theory and the non-parametric analysis based on Varian (1982) and, for the particular case of homotheticity, the contribution of Knoblauch (1993). Section 3 introduces the concept of homothetic efficiency and shows how extended recoverability is still possible when Harp is violated but homothetic efficiency is high. Section 4 uses two data sets to apply the proposed method. Section 5 concludes. The appendix in Section A contains all proofs. A supplementary "computable document" prepared with Wolfram Mathematica® that allows a graphical analysis of revealed preferred and worse sets recovered with our method can be downloaded from the internet (see Section 4.2).

2 PRELIMINARIES

2.1 Notation and Utility Maximisation

The commodity space is $\mathbb{R}^L_+$ and the price space is $\mathbb{R}^L_{++}$, where $L \geq 2$ is the number of different commodities.[4] A (competitive) budget set is defined as $B^i = B(\mathbf{p}^i, w^i) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p}^i\mathbf{x}^i \leq w^i\}$, where $\mathbf{p}^i = (p^i_1, \dots, p^i_L)' \in \mathbb{R}^L_{++}$ is the price vector and $w^i \in \mathbb{R}_{++}$ is the wealth level of the consumer. A demand function $D : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$ of a consumer assigns to each budget set the commodity bundle chosen by the consumer. Unless otherwise noted, we assume that demand satisfies budget balancedness (i.e., $\mathbf{p}^i\mathbf{x}^i = w^i$). We assume that the only observables of the model are $N \geq 1$ different budgets and the corresponding demand of a consumer. It will be convenient to work with normalised prices: If the budget is $B(\mathbf{q}^i, w^i)$, we set $\mathbf{p}^i = \mathbf{q}^i / w^i$.[5] We will then also identify a budget with its characterising price vector, so the entire set of N observations on a consumer is denoted as $\Omega = \{(\mathbf{x}^i, \mathbf{p}^i)\}^N_{i=1}$.

The bundle $\mathbf{x}^i$ is *directly revealed preferred* to a bundle $\mathbf{x}$, written $\mathbf{x}^i \, \mathrm{R}^0 \, \mathbf{x}$, if $\mathbf{p}^i\mathbf{x}^i \geq \mathbf{p}^i\mathbf{x}$; it is *strictly directly revealed preferred* to $\mathbf{x}$, written $\mathbf{x}^i \, \mathrm{P}^0 \, \mathbf{x}$, if $\mathbf{p}^i\mathbf{x}^i > \mathbf{p}^i\mathbf{x}$; it is *revealed preferred* to $\mathbf{x}$, written $\mathbf{x}^i \, \mathrm{R} \, \mathbf{x}$, if R is the transitive closure of R^0, that is, if there exists a sequence $\mathbf{x}^j, \dots, \mathbf{x}^k$, such that $\mathbf{x}^i \, \mathrm{R}^0 \, \mathbf{x}^j \, \mathrm{R}^0 \dots \mathbf{x}^k \, \mathrm{R}^0 \, \mathbf{x}$. The bundle $\mathbf{x}^i$ is *strictly revealed preferred* to $\mathbf{x}$, written $\mathbf{x}^i \, \mathrm{P} \, \mathbf{x}$, if $\mathbf{x}^i \, \mathrm{R} \, \mathbf{x}^j \, \mathrm{P}^0 \, \mathbf{x}^k \, \mathrm{R} \, \mathbf{x}$ for some $j, k = 1, \dots, N$.

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[2] Silva and Stefanou (1996) provided a generalisation of these tests.

[3] For policy matters, the U.S. Congressional Budget Office (CBO), for example, assumes that the economy's underlying production technology is Cobb-Douglas.

[4] The following notation is used: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L$, $\mathbf{x} \geqq \mathbf{y}$ if $x_i \geq y_i$ for all $i = 1, \dots, L$; $\mathbf{x} \geq \mathbf{y}$ if $\mathbf{x} \geqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$ for all $i = 1, \dots, L$. We denote $\mathbb{R}^L_+ = \{\mathbf{x} \in \mathbb{R}^L : \mathbf{x} \geqq (0, \dots, 0)\}$ and $\mathbb{R}^L_{++} = \{\mathbf{x} \in \mathbb{R}^L : \mathbf{x} > (0, \dots, 0)\}$.

[5] This normalisation is routinely applied in revealed preference analysis. The implicit assumption is that demand is homogeneous.

Axiom (Varian 1982) *A set of observations* Ω *satisfies the* Generalised Axiom of Revealed Preference *(*GARP*) if for all* $i, j = 1, \ldots, N$*, it holds that* $[$*not* $\mathbf{x}^i \, \mathrm{P}^0 \, \mathbf{x}^j]$ *whenever* $\mathbf{x}^j \, \mathrm{R} \, \mathbf{x}^i$.

We say that a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalises* a set of observations Ω if $u(\mathbf{x}^i) \geq u(\mathbf{y})$ whenever $\mathbf{x}^i \, \mathrm{R}^0 \, \mathbf{y}$. Let $\mathcal{U}$ denote the set of all continuous, non-satiated, monotonic, and concave utility functions. GARP is easily testable and a necessary and sufficient condition for utility maximisation, as Theorem 1 (Afriat's Theorem) below shows.

Theorem 1 (Afriat 1967, Diewert 1973, Varian 1982, Fostel et al. 2004) *The following conditions are equivalent:*

1. *the set of observations* Ω *satisfies* GARP*;*
2. *there exist numbers* $U^i, \lambda^i > 0$ *such that*

$$U^i \leq U^j + \lambda^j \mathbf{p}^j (\mathbf{x}^i - \mathbf{x}^j) \tag{1}$$

   for all $i, j = 1, \ldots, N$*;*
3. *there exist numbers* V^i *such that*

$$V^i \geq V^j \text{ whenever } \mathbf{p}^i \mathbf{x}^i \geq \mathbf{p}^i \mathbf{x}^j, \text{ and} \tag{2}$$
$$V^i > V^j \text{ whenever } \mathbf{p}^i \mathbf{x}^i > \mathbf{p}^i \mathbf{x}^j \tag{3}$$

   for all $i, j = 1, \ldots, N$*;*
4. *there exists a* $u \in \mathcal{U}$ *which rationalises the set of observations* Ω.

Theorem 1 contains three testable conditions. While conditions 2 and 4 are well-known and can be found in e.g. Varian (1982), condition 3 seem rather new in the literature. Cherchye et al. (2014), for example, use this condition to derive new non-parametric tests for weak separability. We make explicit use of this condition below to formulate new methods to calculate efficiency indices.

To test a set of observations for consistency with GARP, Varian (1982) suggests to use the Floyd-Warshall algorithm (Floyd 1962, Warshall 1962), which is used to find the shortest path from one vertex to another in a weighted graph. To see how this can be used for testing GARP, construct an $N \times N$ matrix $\mathbf{M} = \{m_{i,j}\}$, with

$$m_{i,j} = \begin{cases} 1 & \text{if } \mathbf{x}^i \, \mathrm{R}^0 \, \mathbf{x}^j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can think of $\mathbf{M}$ as a directed graph with N vertices with a path from vertex i to j if $m_{i,j} = 1$. Then construct an $N \times N$ cost matrix $\mathbf{C} = \{c_{i,j}\}$, where

$$c_{i,j} = \begin{cases} 1 & \text{if } m_{i,j} = 1 \\ \infty & \text{otherwise.} \end{cases}$$

$c_{i,j}$ can be interpreted as the cost of moving from vertex i to vertex j, which is finite if and only if there is an edge connecting these to vertices. Applying the Floyd-Warshall algorithm to the cost matrix $\mathbf{C}$ will give us the minimum cost matrix $\mathbf{C}^*$, where $c^*_{i,j} < \infty$ if and only if $\mathbf{x}^i \, \mathrm{R}^0 \, \mathbf{x}^j$. The algorithm therefore allows us to compute the transitive closure of R^0.

However, Varian (1996) already noted that computing the Nth power of the Boolean matrix and applying the signum function to the elements can also be used to compute the transitive closure. This approach can be substantially faster.[6]

Finally, Theorem 1 also shows that one can construct linear programming problems to either check if there exist numbers U^i and λ^j satisfying the Afriat inequalities in Eq. (1), or check whether there exist numbers V^i satisfying the inequalities in Eqs. (2) and (3). The intuition behind these latter inequalities is very simple: if a consumer chooses the bundle $\mathbf{x}^i$ at prices $\mathbf{p}^i$ when $\mathbf{x}^j$ also was affordable, then he gains more utility from consuming $\mathbf{x}^i$, which is reflected by $V^i > V^j$ where V^i can be thought of utility indices at time $i = 1, \ldots, N$.

2.2 *Utility Maximisation and Efficiency*

When a set of observations does not satisfy Garp, it is interesting to obtain a measure of how severe the violation is. One of the most popular measures for the severity of a violation is the *Afriat Efficiency Index* (Aei) due to Afriat (1972), also called the *critical cost efficiency index* (Ccei).[7] Define, for some $e \in [0,1]$, the relation $\mathrm{R}^0(e)$ as $\mathbf{x}^i\, \mathrm{R}^0(e)\, \mathbf{x}^j$ if $e\,\mathbf{p}^i\mathbf{x}^i \geq \mathbf{p}^i\mathbf{x}$, and let $\mathrm{R}(e)$ be the transitive closure of $\mathrm{R}^0(e)$; furthermore, define the relation $\mathrm{P}^0(e)$ as $e\,\mathbf{p}^i\mathbf{x}^i > \mathbf{p}^i\mathbf{x}$. With these concepts, we can define a new version of Garp, called Garp(e).

Axiom *A set of observations Ω satisfies* Garp(e) *for some $e \in (0,1]$ if for all $i, j = 1, \ldots, N$, it holds that* $[$*not* $\mathbf{x}^i\, \mathrm{P}^0(e)\, \mathbf{x}^j]$ *whenever* $\mathbf{x}^j\, \mathrm{R}(e)\, \mathbf{x}^i$.

The Aei is the greatest number e such that Garp(e) is satisfied; it is a measure of wasted income: If a consumer has an Aei of $e < 1$, then he could have obtained the same level of utility by spending only a fraction e of what he actually spent to obtain this level. To compute the Aei when Garp is violated, Varian (1990) describes a binary search routine which he attributes to Houtman and Maks (1987).

The Aei is a summary statistic but does not provide information about which observed choices are causing the deviation from Garp. Varian (1993) defines a more disaggregated measure which he calls the *violation index*. Let $\mathbf{v} = (v_1, \ldots, v_N)$ be a vector, with

$$v_i = \min_{\{j:\, \mathbf{x}^j\, \mathrm{R}\, \mathbf{x}^i\}} \mathbf{p}^i\mathbf{x}^j. \tag{4}$$

If the data satisfy Garp, then $v_i = 1$ for all i. Otherwise, $v_i < 1$ for some i, and this provides information about which $\mathbf{x}^i$ are problematic. A further generalisation of Garp(e) is helpful to understand $\mathbf{v}$.

Axiom *A set of observations Ω satisfies* Garp$(\mathbf{v})$ *for some $\mathbf{v} \in (0,1]^N$ if for all $i, j = 1, \ldots, N$, it holds that*

$$[\textit{not}\ \mathbf{x}^i\, \mathrm{P}^0(v_i)\, \mathbf{x}^j]\ \textit{whenever}\ \mathbf{x}^j\, \mathrm{R}(v_j)\, \mathbf{x}^i.$$

Varian (1993) proves the following proposition.

Proposition 1 (Varian 1993) *Any set of observations Ω satisfies* Garp$(\mathbf{v})$.

[6] The Floyd-Warshall algorithm is of order $O(N^3)$, while the computation of the Nth matrix power can be substantially faster. For example, Coppersmith and Winograd (1990) provide a method that is of order $O(\log_2(K)N^{2.376})$ for the Kth power. Even this can possibly be improved upon for some Boolean matrices (see, e.g., Razzaque et al. 2008).

[7] Reporting the Aei has become a standard for empirical studies, in particular experimental ones. See, for example, Sippel (1997), Mattei (2000), Harbaugh et al. (2001), Andreoni and Miller (2002), Février and Visser (2004), Choi et al. (2007b), Fisman et al. (2007), Dickinson (2009), Camille et al. (2011). See Gross (1995) for a survey of other measures. Most recently, Echenique et al. (2011) provided a new measure based on a money pump argument.

Varian (1993) also notes that the vector **v** does not, in general, give the minimum perturbation of budgets required. He provides an improved violation index, computed with an iterative algorithm that determines the minimal v_i for each i required to break each revealed preference cycle in the data. See also Cox (1997) for a discussion of the improved violation index.

One can also use alternative methods to compute **v**. Motivated by the equivalence between GARP and the Afriat inequalities in Eq. (1), it can be shown that GARP($\mathbf{v}$) is equivalent to there existing numbers U^i and $\lambda^i > 0$ such that the inequalities

$$U^i \leq U^j + \lambda^j \mathbf{p}^j(\mathbf{x}^i - v^j \mathbf{x}^j), \tag{5}$$

hold for all $i, j = 1, \ldots, N$. The indices can then be calculated by maximizing $\|v^1, \ldots, v^N\|$ in some finite dimensional metric $\| \cdot \|$ such that the inequalities in Eq. (5) hold. However, this problem has non-linear (quadratic) constraints which makes it non-trivial, and consequently, may become difficult to solve in practice. Alternatively, the indices can be computed from a slight modification of the inequalities in Eqs. (2) and (3). Specifically, it is easy to see that GARP($\mathbf{v}$) is equivalent to the following inequalities:

$$V^i \geq V^j \text{ whenever } v^i \mathbf{p}^i \mathbf{x}^i \geq \mathbf{p}^i \mathbf{x}^j, \text{ and} \tag{6}$$

$$V^i > V^j \text{ whenever } v^i \mathbf{p}^i \mathbf{x}^i > \mathbf{p}^i \mathbf{x}^j. \tag{7}$$

In contrast to the inequalities in Eq. (5), these inequalities are linear, and therefore more suitable for empirical work. However, since there are unknowns entering both the left-hand and right-hand sides, they cannot be solved with a simple linear programme. To link the two sides, we suggest using binary variables.[8] Specifically, the inequalities in Eqs. (6) and (7) are equivalent to that there exist numbers V^i and binary numbers X^{ij} such that, for all observations $i, j = 1, \ldots, N$,

$$V^i - V^j < X^{ij}, \tag{mip.i}$$

$$(X^{ij} - 1) \leq V^i - V^j, \tag{mip.ii}$$

$$v^i \mathbf{p}^i \mathbf{x}^i - \mathbf{p}^i \mathbf{x}^j < X^{ij} A^i, \tag{mip.iii}$$

$$(X^{ij} - 1) A^j \leq \mathbf{p}^j \mathbf{x}^i - v^j \mathbf{p}^j \mathbf{x}^j, \tag{mip.iv}$$

$$0 \leq V^i < 1, \tag{mip.v}$$

$$X^{ij} \in \{0, 1\}, \tag{mip.vi}$$

where $A^i > \mathbf{p}^i \mathbf{x}^i + 1$ is a fixed number. We suggest to calculate the efficiency indices **v** by solving the following mixed integer linear programming (MILP) problem with respect to V^i, X^{ij} and v^i:

$$\max \sum_{i=1}^{N} v^i \text{ subject to (mip.i)-(mip.vi) and } \mathbf{v} \in (0, 1]^N \tag{8}$$

Since any solution to a MILP problem is, in fact, a global solution, this problem is guaranteed to find a global optimum (in the L1-norm) in the efficiency indices **v**. Given this approach, we can formally define a vector efficiency index: We say that a vector $\tilde{\mathbf{v}}$ is a *Varian Efficiency Vector* (VEV) for Ω if Ω satisfies GARP($\tilde{\mathbf{v}}$) and there does not exist a $\mathbf{v}' \geq \tilde{\mathbf{v}}$ such that Ω satisfies GARP($\mathbf{v}'$). When **v** is computed using the above MILP-approach, it will be a VEV.

[8] See Cherchye et al. (2014) for a similar approach in the context of testing for weak separability of the utility function.

2.3 Recoverability of Preferences

Preferences implicit in a set of data can be recovered using the methods provided by Varian (1982): Given some bundle $\mathbf{x}^0 \in \mathbb{R}^L_+$ which was not necessarily observed as a choice, the set of prices which *support* $\mathbf{x}^0$ is defined as

$$S(\mathbf{x}^0) = \left\{ p^0 \in \mathbb{R}^L_{++} : \ \{(\mathbf{x}^i, \mathbf{p}^i)\}_{i=0}^N \text{ satisfies GARP and } \mathbf{p}^0\mathbf{x}^0 = 1 \right\}. \tag{9}$$

Varian (1982) uses $S(\mathbf{x}^0)$ to describe the set of all bundles which are revealed worse and revealed preferred to a bundle $\mathbf{x}^0$: The set of all bundles which are *revealed worse* than $\mathbf{x}^0$ is given by

$$RW(\mathbf{x}^0) = \left\{ \mathbf{x} \in \mathbb{R}^L_+ : \ \text{for all } \mathbf{p}^0 \in S(\mathbf{x}^0),\ \mathbf{x}^0 \, \mathrm{P} \, \mathbf{x} \right\} \tag{10}$$

and the set of all bundles which are *revealed preferred* to $\mathbf{x}^0$ is given by

$$RP(\mathbf{x}^0) = \left\{ \mathbf{x} \in \mathbb{R}^L_+ : \ \text{for all } \mathbf{p} \in S(\mathbf{y}),\ \mathbf{x} \, \mathrm{P} \, \mathbf{x}^0 \right\}. \tag{11}$$

The following fact follows directly from the definition; see also Varian (1982, Fact 3).

Fact 1 $\mathbf{x} \in RW(\mathbf{x}^0)$ *if and only if* $\mathbf{x}^0 \in RP(\mathbf{x})$.

The *convex hull CH* of a set of points $Y = \left\{ \mathbf{y}^i \right\}_{i=1}^M$ is defined as

$$CH(Y) = \left\{ \mathbf{x} \in \mathbb{R}^L : \mathbf{x} = \sum_{i=1}^M \lambda_i \, \mathbf{y}^i, \mathbf{y}^i \in Y, \lambda_i \geq 0, \sum_{i=1}^M \lambda_i = 1 \right\}. \tag{12}$$

The *convex monotonic hull* of a set of points Y is defined as

$$CMH(Y) = CH\left(\left\{ \mathbf{x} \in X : \mathbf{x} \geq \mathbf{y}^i \text{ for some } i \in \{1, \ldots, M\} \right\} \right). \tag{13}$$

Let $intCMH(Y)$ denote the interior of $CMH(Y)$. The following Proposition provides an easy way of determining whether $\mathbf{x} \in RP(\mathbf{x}^0)$ and, by Fact 1, also whether $\mathbf{x} \in RW(\mathbf{x}^0)$ (see Varian (1982) and Knoblauch (1992) for a proof).

Proposition 2 *Suppose* Ω *satisfies* GARP*. Then*

$$intCMH(\{\mathbf{x}^i : \mathbf{x}^i \, \mathrm{R} \, \mathbf{x}^0\}) \subseteq RP(\mathbf{x}^0) \subseteq CMH(\{\mathbf{x}^i : \mathbf{x}^i \, \mathrm{R} \, \mathbf{x}^0\}).$$

Finally, note that variations of the sets RP and RW can still be constructed if GARP is violated. Obviously, $S(\mathbf{x}^0)$ will be empty in this case, but based on Proposition 2, one can still compute the convex monotonic hulls and analyse the result. However, this will necessarily lead to intersection of $RP(\mathbf{x})$ and $RW(\mathbf{x})$ for some $\mathbf{x}$. Thus, it would more appropriate to base the constructions on $\mathrm{R}(e)$ or $\mathrm{R}(\mathbf{v})$, and to define $S(\mathbf{x}^0)$ in terms of GARP(e) or GARP$(\mathbf{v})$.

2.4 Homotheticity

2.4.1 Definition and Tests

Homotheticity is a restriction on preferences. We say that a utility function is *homothetic* if it is a positive monotonic transformation of a linearly homogeneous utility function; that is, if $u(\mathbf{x}) > u(\mathbf{y})$ then $u(\lambda\mathbf{x}) > u(\lambda\mathbf{y})$ for all $\lambda > 0$. Varian (1983) provides the following axiom, which he shows is equivalent to homothetic rationalisation (Theorem 2).

Axiom (Varian 1983) *A set of observations Ω satisfies the* Homothetic Axiom of Revealed Preference *(Harp) if for all distinct choices of indices $i, j, k, \ldots, \ell$, it holds that*

$$(\mathbf{p}^i\mathbf{x}^j)(\mathbf{p}^j\mathbf{x}^k)\cdots(\mathbf{p}^\ell\mathbf{x}^i) \geq 1.$$

Theorem 2 (Varian 1983) *The following conditions are equivalent:*

1. *the set of observations Ω satisfies Harp;*
2. *there exist numbers $U^i > 0$ such that*

$$U^i \leq U^j\mathbf{p}^j\mathbf{x}^i \tag{14}$$

   for $i, j = 1, \ldots, N$;
3. *there exists a homothetic $u \in \mathcal{U}$ which rationalises the set of observations Ω.*

Varian (1983) provides an efficient method to test a set of observations for consistency with Harp. First note that Harp can equivalently be expressed as

$$\log(\mathbf{p}^i\mathbf{x}^j) + \log(\mathbf{p}^j\mathbf{x}^k) + \ldots + \log(\mathbf{p}^\ell\mathbf{x}^i) \geq 0.$$

One can then apply the Floyd-Warshall algorithm to the cost matrix $\mathbf{C} = \{c_{i,j}\}$ with $c_{i,j} = \log(\mathbf{p}^i\mathbf{x}^j)$. Harp requires that the cost of moving from vertex i to itself in the weighted graph with associated cost matrix $\mathbf{C}$ cannot be made cheaper than 0 (i.e., there are no negative cost cycles). The Floyd-Warshall algorithm will compute the minimum cost for such a graph if there are no negative cost cycles, otherwise it will detect that such a cycle is present.

2.4.2 Homothetic Recoverability of Preferences

Following Knoblauch (1993), define for a set of observations which satisfies Harp a scalar

$$t_{i,\diamond} = \min\left\{(\mathbf{p}^i\mathbf{x}^j)(\mathbf{p}^j\mathbf{x}^k)\cdots(\mathbf{p}^\ell\mathbf{x}^\diamond)\right\}, \tag{15}$$

where the minimum is over all finite sequences $i, j, \ldots, \ell$ between 1 and N inclusive, and $t_{\diamond,\diamond} = 1$. Note that $\diamond$ can be the index of an observed choice $\mathbf{x}^m$ or be equal to 0 for a bundle $\mathbf{x}^0$ which was not observed as a choice. We can compute $t_{i,0}$ for any arbitrary bundle $\mathbf{x}^0$ as we do not need a price vector $\mathbf{p}^0$. We say that $t_{i,\diamond}\mathbf{x}^i$ is *homothetically revealed preferred* to $\mathbf{x}^\diamond$, written $t_{i,\diamond}\mathbf{x}^i \,\mathrm{H}\, \mathbf{x}^\diamond$. The scalar $t = t_{i,\diamond}$ is the smallest value such that $t\,\mathbf{x}^i \,\mathrm{H}\, \mathbf{x}^\diamond$. Note that if $t_{i,\diamond} = (\mathbf{p}^i\mathbf{x}^j)\cdots(\mathbf{p}^k\mathbf{x}^\ell)\cdots(\mathbf{p}^m\mathbf{x}^\diamond)$, then $t_{i,\diamond} = t_{i,k}t_{\ell,\diamond}$.

For practical applications with many different $\mathbf{x}^0$, to compute $t_{i,0}$ it can be more efficient to compute a matrix $\mathbf{T} = \{t_{i,j}\}$ with $i, j = 1, \ldots, N$ once, set $t_{0,0} = 1$, and then find the index j which minimises $t_{i,j}(\mathbf{p}^j\mathbf{x}^0)$. The matrix $\mathbf{T}$ can be computed with the Floyd-Warshall algorithm by setting $c_{i,j} = \log(\mathbf{x}^i\mathbf{p}^j)$; then $t_{i,j} = \exp(c^*_{i,j})$.

Figure 1 illustrates the scalar factors in Eq. (15) with an example with three observations. In (a), we see that $t_{2,3} = \mathbf{p}^2\mathbf{x}^3$. In (b), $\mathbf{x}^1$ can be scaled up so that it still is homothetically revealed preferred to $t_{2,3}\mathbf{x}^2$, and we find that $t_{1,3} = (\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^3) = t_{1,2}t_{2,3}$. This is not a coincidence, as in two dimensions budgets can be sorted by their price ratio. If budgets are sorted and B^1 and B^N have the lowest and highest price ratio, respectively, then $t_{1,N} = (\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^3)\cdots(\mathbf{p}^{N-1}\mathbf{x}^N)$, as was shown in Heufer (2013b).

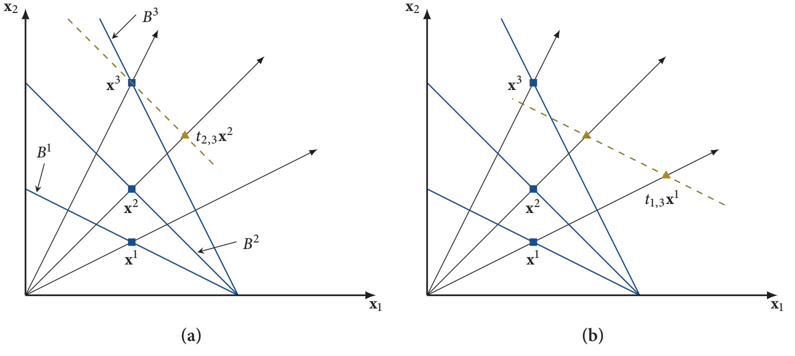

Figure 1: Illustration of the scalar factors.

Knoblauch (1993) also shows how to recover homothetic preferences implicit in a set of observations which satisfies Harp. Define the set of bundles which are *homothetically revealed preferred* to $\mathbf{x}^\diamond$ as

$$HRP(\mathbf{x}^\diamond) = intCMH\left(\mathbf{x}^\diamond \cup \bigcup_{i=0}^{N} t_{i,\diamond}\mathbf{x}^i\right). \tag{16}$$

The set $HRP(\mathbf{x}^\diamond)$ is very useful indeed, as Theorem 3 below shows that it describes the set of bundles which any rationalising homothetic utility function must rank higher than $\mathbf{x}^0$. Define the set of bundles which are *homothetically revealed worse* to $\mathbf{x}^\diamond$ as

$$HRW(\mathbf{x}^\diamond) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{x}^\diamond \in HRP(\mathbf{x})\}. \tag{17}$$

As HRP can be easily computed as the convex hull of a finite number of points, it is also easy to test for any bundle if $\mathbf{x} \in HRW(\mathbf{x}^\diamond)$.

Theorem 3 (Knoblauch 1993) *Suppose Ω satisfies Harp. The following conditions are equivalent:*

1. $\mathbf{x} \in HRP(\mathbf{x}^\diamond)$;
2. *every homothetic $u \in \mathcal{U}$ which rationalises Ω satisfies $u(\mathbf{x}) > u(\mathbf{x}^\diamond)$.*

See Knoblauch (1993) for the proof. The following corollary is then straightforward, and we omit the proof.

Corollary 1 *Suppose Ω satisfies Harp. The following conditions are equivalent:*

1. $\mathbf{x} \in HRW(\mathbf{x}^\diamond)$;
2. *every homothetic $u \in \mathcal{U}$ which rationalises Ω satisfies $u(\mathbf{x}^\diamond) > u(\mathbf{x})$.*

It should also be obvious that $RP(\mathbf{x}^\diamond) \subseteq HRP(\mathbf{x}^\diamond)$ and $RW(\mathbf{x}^\diamond) \subseteq HRW(\mathbf{x}^\diamond)$ for all $\mathbf{x}^\diamond$.

3 HOMOTHETIC EFFICIENCY

3.1 *A Lower Bound on Homothetic Efficiency and Recoverability*

3.1.1 *Homothetic Afriat Efficiency Index*

Suppose we have a set $\Omega = \{(\mathbf{x}^i, \mathbf{p}^i)\}_{i=1}^2$. If the consumer had homothetic preferences, then his demand when facing budget $B(\mathbf{p}^1, t)$ would be $t\,\mathbf{x}^1$. Then the smallest t for which $t\,\mathbf{x}^1$ would be revealed preferred to $\mathbf{x}^2$ is $t = \mathbf{p}^1\mathbf{x}^2$. Now suppose Ω does not satisfy Harp such that $(\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^1) < 1$. Then the choice $(\mathbf{p}^1\mathbf{x}^2)\mathbf{x}^1$ on $B(\mathbf{p}^1, \mathbf{p}^1\mathbf{x}^2)$ would be revealed preferred to $\mathbf{x}^2$, but as Harp is violated, $\mathbf{p}^2\mathbf{x}^2 = 1 > [\mathbf{p}^2(\mathbf{p}^1\mathbf{x}^2)\mathbf{x}^1] = (\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^1)$, that is, $\mathbf{x}^2$ is strictly revealed preferred to $(\mathbf{p}^1\mathbf{x}^2)\mathbf{x}^1$, which would violate Garp. But if we relax Garp, as is done to compute the Aei, we can find the greatest $e \in (0,1]$ such that $e\,\mathbf{p}^2\mathbf{x}^2 = e \leq [\mathbf{p}^2(\mathbf{p}^1\mathbf{x}^2)\mathbf{x}^1]$, that is, $e = (\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^1)$. This e has a similar economic interpretation as the Aei: If the preferences of the consumer were homothetic but demand is specified with errors, then e can be interpreted as a level of expenditure which the consumer wasted due to the errors; in particular, he could have obtained the same utility as he obtained from choosing $\mathbf{x}^2$ at an expenditure $e < 1$ rather than the actual expenditure of 1.

However, if we use this approach, the multiplication of the scalars $\mathbf{p}^i\mathbf{x}^j$ can lead to very low values of e. For example, in the experiment conducted by Fisman et al. (2007), subjects were asked to choose bundles from fifty budgets. They were not required to spend their entire wealth, which lead to some minor violations of budget balancedness. If the choices are evaluated by taking the difference between $\mathbf{p}^i\mathbf{x}^i$ and w^i into account, then repeated multiplication can distort the result. To illustrate the general problem with a hypothetical scenario, suppose that a subject is asked to make ten choices from the same budget set. Suppose we observe $\mathbf{x}^i = \mathbf{x}^j$ for all $i, j = 1, \ldots, N$, with $\mathbf{p}^i = \mathbf{p}^j$ for all $i, j = 1, \ldots, N$ as well. Suppose that $\mathbf{p}^i\mathbf{x}^i = .95$, but $w^i = 1$. Then $(\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^3)\cdots(\mathbf{p}^{10}\mathbf{x}^1) \approx 0.5987$, even though only five percent of the wealth level was wasted each time and even though the data would satisfy Harp if the w^i were set to .95. But even without violations of budget balancedness and with different budgets, minor errors can lead to very low values of e if many choices are observed.

We therefore suggest to use the following axiom, called Harp(e), which takes into account the number of scalars which are multiplied.

Axiom *A set of observations Ω satisfies* Harp(e) *for some $e \in (0,1]$ if for all distinct choices of indices $i, j, k, \ldots, \ell$, it holds that*

$$\left(\frac{\mathbf{p}^i\mathbf{x}^j}{e}\right)\left(\frac{\mathbf{p}^j\mathbf{x}^k}{e}\right)\cdots\left(\frac{\mathbf{p}^\ell\mathbf{x}^i}{e}\right) \geq 1.$$

Figure 2 illustrates the idea. Figure 2.(a) shows the two observations. The dashed line shows the boundary of the shifted budget B^1 which contains $\mathbf{x}^2$. The intersection of the dashed line and the ray through the origin and $\mathbf{x}^1$, shown as $\lambda\mathbf{x}^1$, gives the demand on the shifted budget if preferences were homothetic. Here λ is chosen to equal $\mathbf{p}^1\mathbf{x}^2$; note that, by Theorem 3, $\lambda\mathbf{x}^1$ would be homothetically revealed preferred to $\mathbf{x}^2$ if preferences were homothetic. But as $\lambda\mathbf{x}^1$ is in the interior of budget B^2, $\mathbf{x}^2$ is strictly revealed preferred to $\lambda\mathbf{x}^1$, thus Harp is violated. Also note that there is a $\mu < 1$ such that $\mu\mathbf{x}^2$ would be homothetically revealed preferred to $\lambda\mathbf{x}^1$. Then $\tilde{\lambda}\mathbf{x}^1$ with $\tilde{\lambda} < \lambda$ would be homothetically revealed preferred to $\mu\mathbf{x}^2$. This process can be repeated ad infinitum.

Figure 2.(b) shows the two budgets shifted downwards by setting the wealth level to $e < 1$. Figure 2.(c) shows that if $\mathbf{x}^1$ is scaled upwards by a factor equal to λ/e, we find that while $\mathbf{x}^2$ is still strictly revealed preferred to it — $\mathbf{x}^2\ \mathrm{P}^0\ (\lambda\mathbf{x}^1/e)$ — it is not strictly revealed preferred at efficiency level e — [not $\mathbf{x}^2\ \mathrm{P}^0(e)\ (\lambda\mathbf{x}^1/e)$]. This

is indeed the smallest e for which Harp(e) is satisfied; as the example has only two observations, this e can be found by setting $[(\mathbf{p}^1\mathbf{x}^2)/e][(\mathbf{p}^2\mathbf{x}^1)/e] = 1$ and solving for e.

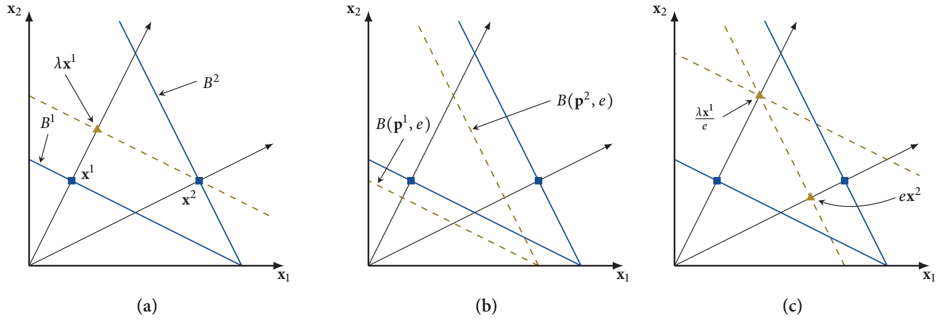

Figure 2: An illustration of Harp(e) and the homothetic efficiency index. The example uses $\mathbf{x}^1 = (2,4)$, $\mathbf{p}^1 = (1/10, 1/5)$, $\mathbf{x}^2 = (8,4)$, and $\mathbf{p}^2 = (1/10, 1/20)$. The greatest e for which Harp(e) is satisfied is $4/5$.

Given Harp(e), we propose the following definition, in analogy to the Aei:

Definition *For a set of observations* Ω*, the* Homothetic Efficiency Index *(Hei) is the greatest* $e \in (0,1]$ *such that* Ω *satisfies Harp*(e).

The Hei can be computed using the same binary search algorithm used for calculating the Aei. It can be reported as a summary statistic. The next theorem will provide a good motivation and justification to compute and report e. It is based on a concept we call e-rationalisation, which is in the same spirit as the definitions proposed and analysed by Halevy et al. (2012).

Definition *A utility function* $u \in \mathcal{U}$ e-rationalises *a set of observations* Ω *if* $u(\mathbf{x}^i) \geq u(\mathbf{y})$ *whenever* $\mathbf{x}^i \, \mathrm{R}^0(e) \, \mathbf{y}$.

For e close to 1, a utility function which e-rationalises a set of data still adequately explain choices as the result of utility maximisation with minor errors. The following theorem shows that Harp(e) is necessary and sufficient for homothetic e-rationalisation.

Theorem 4 *For any* $e \in [0,1]$ *the following conditions are equivalent:*

1. *the set of observations* Ω *satisfies Harp*(e)*;*
2. *there exist numbers* $U^i > 0$ *such that*

$$e\,U^i \leq U^j \mathbf{p}^j \mathbf{x}^i \tag{18}$$

   for $i, j = 1, \ldots, N$*;*

3. *there exists a homothetic* $u \in \mathcal{U}$ *which* e*-rationalises the set of observations* Ω.

As discussed above, one alternative is to compute Hei from Harp(e) using a simple binary search algorithm; another alternative is to set up a linear programming problem based on Eq. (18) to calculate the maximal e. Since $e\,U^i > 0$ we can (log-)linearise Eq. (18) to get

$$\epsilon + u^i - u^j \leq \log(\mathbf{p}^j\mathbf{x}^i), \tag{19}$$

for all $i, j = 1, \ldots, N$, where $u^i = \log(U^i)$, $u^j = \log(U^j)$ and $\epsilon = \log(e)$. The maximal efficiency index can then be computed as (with respect to $\epsilon \in (-\infty, 0]$ and $u^i \in (-\infty, \infty)$ for all $i = 1, \ldots, N$):

$$\max \epsilon \text{ subject to (19).} \tag{20}$$

This is a linear programme and can therefore be solved in polynomial time. In section 3.2.1 we discuss the properties of this problem in more detail and provide a motivation for its use in applied work.

3.1.2 *Recoverability with Homothetic Efficiency*

Homothetic recoverability with data adjusted by efficiency indices is not as straightforward in the standard case described in Section 2.3. It is not sufficient to divide the scalar factors $t_{i,\diamond}$ in Eq. (15) by e, or to divide each component in Eq. (15) by e to find new scalar factors. Figure 2.(c) already illustrates this: With only two observations, we would have $t_{1,2} = \mathbf{p}^1\mathbf{x}^2$ if the data satisfied HARP. But here, $\lambda = \mathbf{p}^1\mathbf{x}^2$, and clearly, $\mathbf{x}^2$ is still revealed preferred to $\lambda\mathbf{x}^1/e$. We need to scale down $\mathbf{x}^1$ by e as well to remove this contradiction, so $\lambda\mathbf{x}^1/e$ is only homothetically revealed preferred to $e\mathbf{x}^2$. Alternatively, $\lambda\mathbf{x}^1/e^2$ would be homothetically revealed preferred to $\mathbf{x}^2$ because $\mathbf{p}^2(\lambda\mathbf{x}^1/e^2) = 1$.

Let $\hat{e}$ be the HEI of a set of observations Ω. Define

$$\hat{t}_{i,\diamond} = \min\left\{\left(\frac{\mathbf{p}^i\mathbf{x}^j}{\hat{e}^2}\right)\left(\frac{\mathbf{p}^j\mathbf{x}^k}{\hat{e}^2}\right)\cdots\left(\frac{\mathbf{p}^\ell\mathbf{x}^\diamond}{\hat{e}^2}\right)\right\}, \tag{21}$$

where the minimum is over all finite sequences $i, j, \ldots, \ell$ between 1 and N inclusive. We say that $\hat{t}_{i,\diamond}\mathbf{x}^i$ is *homothetically revealed preferred at efficiency level* $\hat{e}$ to $\mathbf{x}^\diamond$, written $\hat{t}_{i,\diamond}\mathbf{x}^i\,\hat{\mathrm{H}}\,\mathbf{x}^\diamond$. To motivate this definition and show that it is useful, we first need to define the set of bundles which are homothetically revealed preferred at efficiency level $\hat{e}$ to $\mathbf{x}^\diamond$ as

$$\widehat{HRP}_{\hat{e}}(\mathbf{x}^\diamond) = intCMH\left(\mathbf{x}^\diamond \cup \bigcup_{i=1}^{N} \hat{t}_{i,\diamond}\mathbf{x}^i\right), \tag{22}$$

and the worse set as

$$\widehat{HRW}_{\hat{e}}(\mathbf{x}^\diamond) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{x}^\diamond \in \widehat{HRP}_{\hat{e}}(\mathbf{x})\}. \tag{23}$$

Figure 3.(a) shows an example of the sets $\widehat{HRP}_{\hat{e}}(\mathbf{x}^0)$ and $\widehat{HRW}_{\hat{e}}(\mathbf{x}^0)$ (hatched area) based on the HEI, together with the standard sets $RP(\mathbf{x}^0)$ and $RW(\mathbf{x}^0)$ from Section 2.3. Figure 3.(b) shows the boundary of the set $\widehat{HRP}(\hat{e})$ for the two observations and two bundles in between.

Ideally, $\widehat{HRP}$ is an extension of RP, that is, $RP(x^0) \subseteq HRP_e(x^0)$, and the same for $\widehat{HRW}$ and RW. We could then claim that $\widehat{HRP}$ was faithful to the revealed preference relation and recovers additional information about a consumer's preference based on the assumption that preferences are indeed homothetic, but homotheticity was accidentally violated. However, this is not always the case, as a simple example demonstrates. Suppose $\mathbf{x}^1 = (2, 4)$, $\mathbf{x}^2 = (5, 5)$, $\mathbf{x}^3 = (8, 4)$, $\mathbf{p}^1 = (1/10, 1/5)$, $\mathbf{p}^2 = (1/10, 1/10)$, and $\mathbf{p}^3 = (1/10, 1/20)$. Homotheticity is violated, and the HEI is 4/5. We then find that $\hat{t}_{3,2} \approx 1.172$, and therefore $\hat{t}_{3,2}\mathbf{x}^3 > \mathbf{x}^3$. But $\mathbf{x}^3 \, \mathrm{R}^0 \, \mathbf{x}^2$, so $RP(\mathbf{x}^2)$ contains $\mathbf{x}^3$, but $\widehat{HRP}(\mathbf{x}^2)$ does not.

This problem becomes smaller as $\hat{e}$ gets closer to 1, but we still recommend to use the HEI mostly as a summary statistic. The reason is that there is a better way to adjust data for violations of homotheticity as we will show in the next section, which leads to superior versions of $\widehat{HRP}$ and $\widehat{HRW}$ for applications. However, the next theorem does provide the motivation for using the sets $\widehat{HRP}$ and $\widehat{HRW}$.

While Theorem 4 provides the motivation to report the HEI as a summary statistic, Theorem 5 below justifies the use of $\widehat{HRP}$ and $\widehat{HRW}$. It shows that every homothetic utility function which e-rationalises the data will agree with our construction of the homothetically revealed preferred and worse sets.

Theorem 5 *If Ω satisfies HARP(e), then for every homothetic $u \in \mathcal{U}$ which e-rationalises Ω,*

$$\widehat{HRP}_{\hat{e}}(\mathbf{x}^\diamond) \subseteq \left\{\mathbf{x} \in \mathbb{R}^L_+ : u(\mathbf{x}) \geq u(\mathbf{x}^\diamond)\right\},$$
$$\widehat{HRW}_{\hat{e}}(\mathbf{x}^\diamond) \subseteq \left\{\mathbf{x} \in \mathbb{R}^L_+ : u(\mathbf{x}) \leq u(\mathbf{x}^\diamond)\right\}.$$

3.2 *Improved Homothetic Efficiency Vector and Recoverability*

3.2.1 *Definition and Computation*

Similar to the case of the AEI and Varian's (1993) improved violation index, the HEI is only a lower bound on homothetic efficiency. A *homothetic efficiency vector* which provides information about how much each budget has to be perturbed to achieve a meaningful kind of consistency while keeping the perturbations minimal would be informative and useful for applied work. We suggest the following straightforward generalisation of HARP(e).

Axiom *A set of observations Ω satisfies HARP$(\mathbf{h})$ for some $\mathbf{h} = (h_1, \ldots, h_N) \in (0,1]^N$ if for all $i, j = 1, \ldots, N$, it holds that*

$$\left(\frac{\mathbf{p}^i\mathbf{x}^j}{h_i}\right)\left(\frac{\mathbf{p}^j\mathbf{x}^k}{h_j}\right)\cdots\left(\frac{\mathbf{p}^\ell\mathbf{x}^i}{h_\ell}\right) \geq 1.$$

The problem with computing a vector $\mathbf{h}$ with maximal values is that "breaking cycles" is not as easy as in the standard case in Varian (1993). It is not feasible to consider breaking "homothetically revealed preference cycles". If a set of data does not satisfy HARP, then Knoblauch's (1993) concept of "homothetically revealed preferred to" is either ill defined or computationally infeasible for many observations. If in the definition of the scalar factors in Eq. (15) we allow for multiple occurrences of indices, then the minimum is not defined and the infimum is 0. For example, if $(\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^1) < 1$, then $\lim_{n\to\infty}[(\mathbf{p}^1\mathbf{x}^2)(\mathbf{p}^2\mathbf{x}^1)]^n = 0$. If we restrict Eq. (15) to distinct vertices, then the problem of computing the scalars amounts to the NP-hard problem of finding a simple shortest path in a weighted complete graph (i.e., a path that visits each vertex at most once, except for the first vertex if the path is a cycle). The complexity of this endeavour quickly approaches a level which makes computation infeasible.[9]

[9] In a complete graph with $N \geq 2$ vertices, there are $\sum_{i=2}^N \frac{(N-2)!}{(N-i)!}$ different simple paths between two distinct vertices. For the 50 observations per subject collected in Fisman et al. (2007) and Choi et al. (2007a), this would require to compare $3.37445 \cdot 10^{61}$ different paths.

We will therefore rely on a linear programming approach. The basis for this approach will be provided by a theorem which is the analogue of Theorem 4 (the proof is practically the same as for Theorem 4 and we omit it.)

Definition *A utility function* $u \in \mathcal{U}$ **h**-rationalises *a set of observations* Ω *if* $u(\mathbf{x}^i) \geq u(\mathbf{y})$ *whenever* $\mathbf{x}^i \, \mathrm{R}^0(h_i) \, \mathbf{y}$.

Theorem 6 *For any* $\mathbf{h} = (h_1, \ldots, h_N) \in [0,1]^N$ *the following conditions are equivalent:*

1. *the set of observations* Ω *satisfies* Harp($\mathbf{h}$);
2. *there exist numbers* $U^i > 0$ *such that:*

$$h_j \, U^i \leq U^j \mathbf{p}^j \mathbf{x}^i \tag{24}$$

for $i, j = 1, \ldots, N$;

3. *there exists a homothetic* $u \in \mathcal{U}$ *which* **h**-*rationalises the set of observations* Ω.

We say that $\tilde{\mathbf{h}}$ is a *Homothetic Efficiency Vector* (Hev) for Ω if Ω satisfies Harp($\tilde{\mathbf{h}}$) and there does not exist a $\mathbf{h}' \geq \tilde{\mathbf{h}}$ such that Ω satisfies Harp($\mathbf{h}'$).

As discussed above, calculating Hev can be computationally difficult. For example, computing the Hev as close as possible to the unit vector in the Minkowski distance norm amounts to solving the following problem:

$$\max \left(\sum_{i=1}^{N} (h_i - 1)^{\varphi} \right)^{1/\varphi} \text{ such that either Harp}(\mathbf{h}) \text{ or Eq. (24) holds.} \tag{25}$$

Although this problem is NP-hard for any $\varphi \geq 1$, it is no longer NP-hard for $\varphi = 0$, in which case the objective function reduces to $\lim_{\varphi \to 0} (\sum (h_i - 1)^{\varphi})^{1/\varphi} = \prod h_i$. Thus, the problem:

$$\max \prod_{i=1}^{N} h_i \text{ such that either Harp}(\mathbf{h}) \text{ or Eq. (24) holds,} \tag{26}$$

can be solved in polynomial time. To understand why, simply note that the problem is invariant to any strictly monotonic transformation of the objective function. Thus, by a log-transform we can replace the objective function in (26) with $\sum_{i=1}^{N} \kappa_i$, where $\kappa_i = \log(h_i)$. This results in the linear programme (solved with respect to $\kappa_i \in (-\infty, 0]$ and $u^i \in (-\infty, \infty)$ for all $i = 1, \ldots, N$):

$$\max \sum_{i=1}^{N} \kappa_i \text{ subject to } \kappa_j + u^i - u^j \leq \log(\mathbf{p}^j \mathbf{x}^i), \tag{27}$$

for $i, j = 1, \ldots, N$, with $u^i = \log(U^i)$ and where the constraints in (27) are log-linearisations of the inequalities in Eq. (24). This programme can be solved using elementary linear programming techniques, and consequently in polynomial time.

Although strictly speaking, the problem (27) does not compute the set of indices closest to the unit vector in a 'true' norm (since the Minkowski metric with $\varphi = 0$ does not technically satisfy all requirements for a norm), it is a commonly used procedure in practice to find maximal elements in constrained optimisation problems. However, it can be shown that the indices calculated from the problem (27) are first order approximations of the indices calculated under $\varphi = 1$ in the general problem (25). Indeed, a first order Taylor approximation of $\log(h)$ about the point 1 yields $\log(h) \simeq (h - 1)$. Thus, the objective function (27) satisfies $\sum_{i=1}^{N} \kappa_i = \sum_{i=1}^{N} \log(h_i) \simeq \sum_{i=1}^{N} (h_i - 1)$, where the right-hand side then corresponds to (25) evaluated in $\varphi = 1$.

3.2.2 *Recoverability with Improved Homothetic Efficiency*

As in Section 3.1.2, we can define scalar factors for homothetically revealed preferred relations. Based on that, we can again construct homothetically revealed preferred and worse sets.

Let $\tilde{\mathbf{h}}$ be an HEV of a set of observations Ω. Define

$$\tilde{t}_{i,m} = \min\left\{\left(\frac{\mathbf{p}^i\mathbf{x}^j}{h_i^2}\right)\left(\frac{\mathbf{p}^j\mathbf{x}^k}{h_j^2}\right)\cdots\left(\frac{\mathbf{p}^\ell\mathbf{x}^m}{h_\ell^2}\right)\right\}, \tag{28}$$

where the minimum is over all finite sequences $i, j, \ldots, \ell$ between 1 and N inclusive. Again, we say that $\tilde{t}_{i,\diamond}\mathbf{x}^i$ is *homothetically revealed preferred at efficiency level* h_i to $\mathbf{x}^\diamond$, written $\hat{t}_{i,\diamond}\mathbf{x}^i\, \tilde{\mathrm{H}}\, \mathbf{x}^\diamond$. Define

$$\widetilde{HRP}_{\tilde{\mathbf{h}}}(\mathbf{x}^\diamond) = intCMH\left(\mathbf{x}^\diamond \cup \bigcup_{i=1}^{N} \tilde{t}_{i,\diamond}\mathbf{x}^i\right), \tag{29}$$

and

$$\widetilde{HRW}_{\tilde{\mathbf{h}}}(\mathbf{x}^\diamond) = \{\mathbf{x} \in \mathbb{R}_+^L : \mathbf{x}^\diamond \in \widetilde{HRP}_{\tilde{\mathbf{h}}}(\mathbf{x})\}. \tag{30}$$

Figure 3.(c) shows an example of the sets $\widetilde{HRP}_{\tilde{\mathbf{h}}}(\mathbf{x}^0)$ and $\widetilde{HRW}_{\tilde{\mathbf{h}}}(\mathbf{x}^0)$ based on an HEV with $\tilde{\mathbf{h}} = (1, 16/25)$. This $\tilde{\mathbf{h}}$ maximises the sum of h_1 and h_2, that is, there is no HEV $\mathbf{h}'$ with $\mathbf{h}' \geq \tilde{\mathbf{h}}$ such that $h_1' + h_2' > 1 + 16/25$. Figure 3.(d) shows same for an alternative HEV which maximises $h_1 h_2$.

The next theorem motivates these definitions (again the proof of Theorem 7 is practically the same as for Theorem 5 and we omit it).

Theorem 7 *If Ω satisfies* HARP$(\mathbf{h})$, *then for every homothetic* $u \in \mathcal{U}$ *which e-rationalises Ω,*

$$\widetilde{HRP}_{\tilde{\mathbf{h}}}(\mathbf{x}^\diamond) \subseteq \left\{\mathbf{x} \in \mathbb{R}_+^L : u(\mathbf{x}) \geq u(\mathbf{x}^\diamond)\right\},$$
$$\widetilde{HRW}_{\tilde{\mathbf{h}}}(\mathbf{x}^\diamond) \subseteq \left\{\mathbf{x} \in \mathbb{R}_+^L : u(\mathbf{x}) \leq u(\mathbf{x}^\diamond)\right\}.$$

4 APPLICATIONS

In this section, we apply our methods to one experimental data set and one survey data set. Our aim with this empirical exercise is threefold: first, we want to show that our methods can recover detailed information about subjects' preferences in experimental data sets. Second, we want to show that homothetic efficiency can be high for consumer choice data. Finally, we want to show that data which is adjusted by HEV (HEI) can have much more discriminatory power against irrational behaviour than VEV (AEI) adjusted data.

4.1 *Test Power and Conditional Test Power*

The standard approach to calculate the power for revealed preference tests is based on Bronars' (1987). In this paper, we follow Bronars' approach and generate many random choice sets uniformly distributed on the budgets and compute the fraction of sets that either violates GARP or HARP. We refer to the fraction of generated data sets violating GARP and HARP as the power of GARP and HARP, respectively. Moreover, to analyse the loss in power for income adjusted data, we employ the following three-step procedure: (i) we compute efficiency indices from the observed data; (ii) then we generate random data sets using Bronars' approach, and (iii)

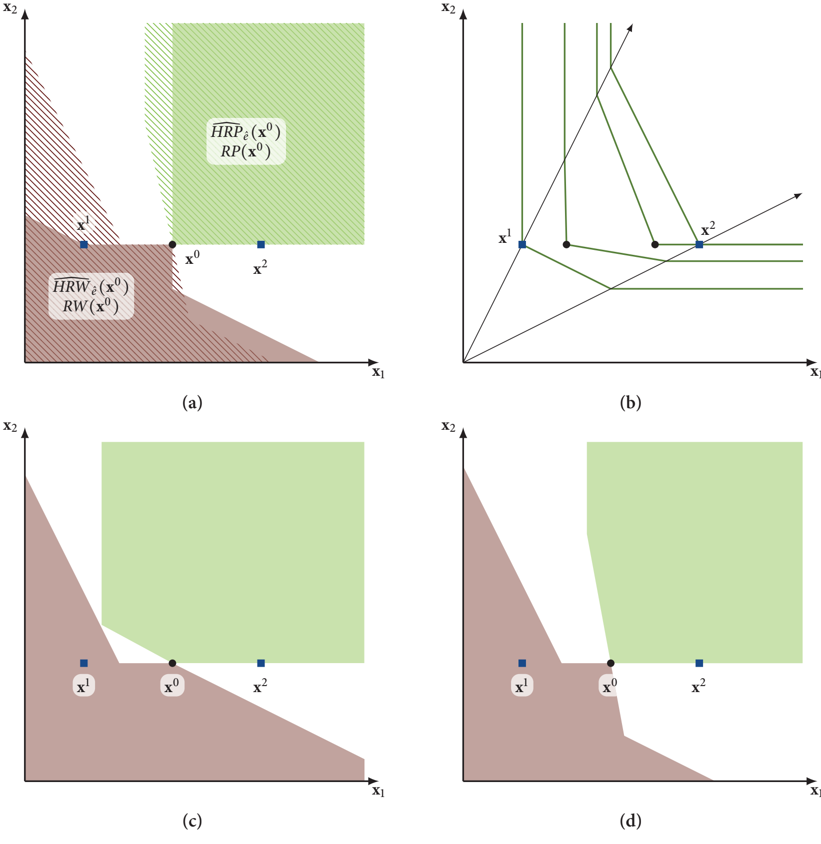

Figure 3: (a) (Homothetically) revealed preferred and worse sets for $\mathbf{x}^0 = (\mathbf{x}^1 + \mathbf{x}^2)/2$ with the same data as in Figure , based on $\hat{e} = 4/5$. The hatched area shows $\widehat{HRP}$ and $\widehat{HRW}$, the filled area RP and RW. (b) The boundary of $\widehat{HRP}$ for $\mathbf{x}^1$, $\mathbf{x}^2$, and two bundles in between, again with $\hat{e} = 4/5$. (c) $\widehat{HRP}$ and $\widehat{HRW}$ based on a Hev with $\tilde{\mathbf{h}} = (1, 16/25)$ which maximises $h_1 + h_2$ and $h_1 h_2$. (d) $\widehat{HRP}$ and $\widehat{HRW}$ based on an Hev with $\tilde{\mathbf{h}} = (256/325, 13/16)$ which also maximises $h_1 h_2$ but not $h_1 + h_2$.

finally, we calculate the fraction of sets violating either Garp or Harp where total expenditure (income) is adjusted for efficiency; that is, in this final step, we deflate expenditure by the efficiency index computed in the first step when running Garp and Harp. Repeating the three-step procedure for all four efficiency indices Aei, Vev, Hei, and Hev allows us to compare the loss in discriminatory power across the indices.

A potential issue with calculating the power for Harp is that Garp is a necessary condition for Harp, and Hei can never exceed Aei. It would therefore be interesting to know the probability that a set of random choices which happens to satisfy Garp also satisfies Harp, or that such a set has at least the same Hei. This would provide the homotheticity test power *conditional* on Garp being satisfied. Heufer (2013a) provides an efficient method to generate sets of random choices which satisfy Garp. We can then test these sets for Harp which we refer to as the conditional test power. We do this for the data analysed below.

4.2 Experimental Data: Preferences for Giving

Fisman et al. (2007, FKM) analyse data obtained in a laboratory experiment. They employ the same setup as Andreoni and Miller (2002, AM), that is, a generalised dictator game in which one subject (the dictator) allocates token endowments between himself and an anonymous other subject with different transfer rates. The payoffs of the dictator and the beneficiary are interpreted as two distinct goods, and the transfer rates as the price ratio. In both papers, the authors estimate a CES utility function, so they implicitly maintain the hypothesis that choices are homothetic. Testing how "close" the choices are to homotheticity is therefore important and should be conducted at least as a pretest to screen out particularly inefficient choices.

A simple two-dimensional version of homothetic efficiency has been computed for both the FKM and the AM data by Heufer (2013b). We only focus on the FKM data here, as they contain 50 choices per subject as opposed to 8 in the AM data. This also allows for an informative graphical analysis based on the sets $\widetilde{HRP}$ and $\widetilde{HRW}$.

We start our analysis by calculating Aei, Hei, Vev and Hev for all subjects.[10] These results are presented in Table 1, where the two first rows report the mean and minimum, the first, second (median) and third quantiles and the maximum of Aei and Hei calculated across all 76 subjects. As expected, we find that Aei is noticably higher than Hei for most subjects. However, as discussed above, Aei and Hei are summary statistics, and may therefore be uninformative in describing the entire distribution of the indices. In fact, looking at the third and fourth rows of Table 1, which report summary statistics for Vev and Hev, gives a different picture. These results show that homothetic efficiency is in fact close to utility maximisation efficiency.[11] Specifically, Hev displays the same pattern as Vev: both indices are characterised by one or a few observations with lower efficiency values, while the remaining values in the vector are very close to one.

Table 2 report summary statistics for the unconditional power calculations. We present the power of Garp and Harp for different configurations, depending on how we adjust total expenditure when applying these tests (see the second column). The results from the last two rows show that Garp and Harp have optimal power even when total expenditure is deflated by Vev and Hev, respectively. Thus, Garp and Harp have high discriminatory power even for income-adjusted random data. However, as seen from the third and fourth row, this is not always the case when deflating expenditure by Aei and Hei. Specifically, the loss in power can be

[10] For each subject, Hei and Hev are computed by solving problems (20) and (27), respectively. To aid comparisons with Aei and Vev we calculate these indices from a slight modification of problem (8). Specifically, we transform the problem so that the indices are computed in the L0-norm by log-linearising the constraints (mip.i)-(mip.vi) and then maximize $\sum_{i=1}^{N} v^i$ where $v^i = \log(v^i)$. We also computed Aei and Vev by solving the problem (8). Interestingly, computing the indices in the L0 and L1 norms gave practically identical solutions. This suggests that the L0-problem is a very good approximation to the L1-problem (See the discussion in Section 3.2.1.

[11] The entries in rows 3 and 4 are averages across all subjects. For example, to obtain the entry minimum we first computed $\min\{v_1, \ldots, v_N\}$ and $\min\{h_1, \ldots, h_N\}$ for each subject and then calculated the mean of these values over all subjects.

Efficiency						
Index	mean	min	1st quantile	median	3rd quantile	max
Aei	0.9407	0.5308	0.9173	0.9775	0.9972	1.0000
Hei	0.8821	0.4438	0.8118	0.9210	0.9786	1.0000
Vev	0.9959	0.9350	0.9982	1.0000	1.0000	1.0000
Hev	0.9788	0.8274	0.9742	0.9919	0.9986	1.0000

Table 1: Efficiency indices (FKM).

rather considerable for some subjects as revealed by the fourth column in rows 3 and 4. Finally, Table 3 reports the conditional power results. We find that Harp has optimal power against uniformly random data which satisfies Garp. Next, we go on to show how our methods can recover detailed information about subjects' preferences.

Unconditional power							
Axiom	Income deflated	mean	min	1st quantile	median	3rd quantile	max
Garp	No	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Harp	No	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Garp	Aei	0.9759	0.2520	1.0000	1.0000	1.0000	1.0000
Harp	Hei	0.9810	0.3720	1.0000	1.0000	1.0000	1.0000
Garp	Vev	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Harp	Hev	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: Unconditional power for Garp and Harp (FKM).

Conditional power						
Axiom	mean	min	1st quantile	median	3rd quantile	max
Harp	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3: Conditional power for Harp (FKM).

Figure 4 shows examples of subjects who reveal almost prototypical preferences. The differences between the homothetically revealed preferred and worse sets and the regular revealed preferred and worse sets demonstrate how much more we can learn about subjects' preferences when deviations from homotheticity are minor. The examples also illustrate that the theoretical problem of disagreements between *RP* and *HRP* described in Section 3.1.2 is unlikely to occur for large sets of real data.

We also provide an interactive application prepared with Wolfram Mathematica® that allows users to analyse the data graphically and create figures as the ones in Figure 4 for arbitrary subjects and bundles. This software is available on one of the authors' websites and can be run with the free Wolfram CDF Player.[12]

[12] The CDF file is available at `https://sites.google.com/site/janheufer/HomotheticRecoverability.cdf`. To run the file, the free Wolfram CDF Player can be obtained at `http://www.wolfram.com/cdf-player/`.

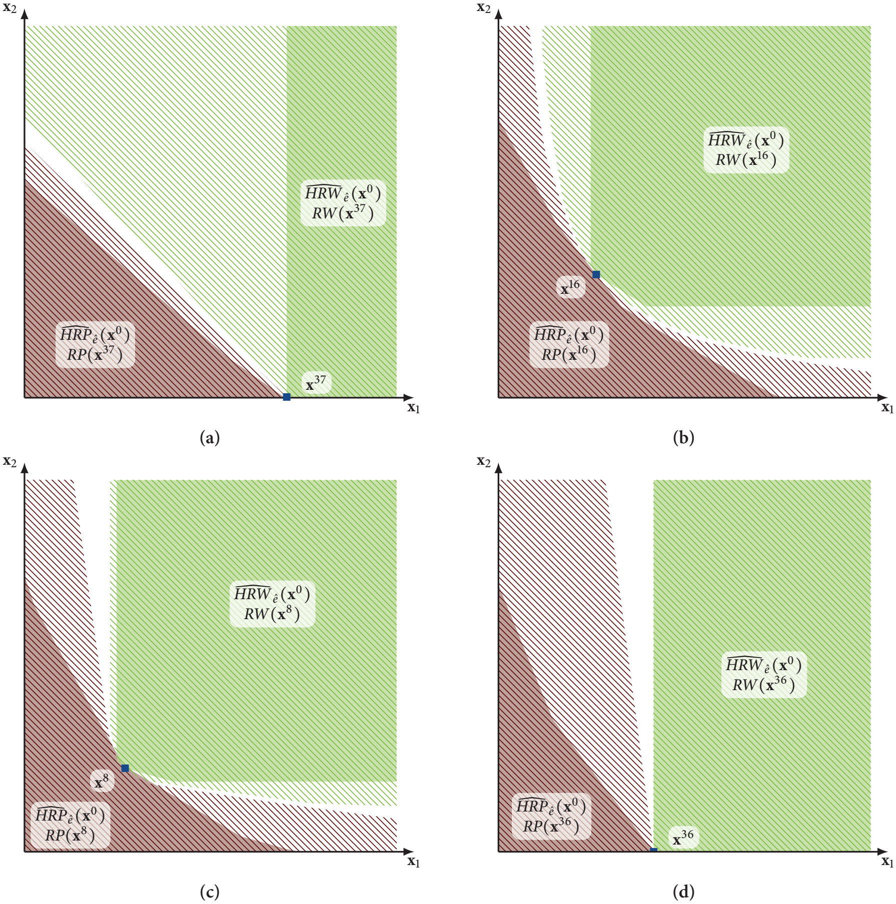

Figure 4: The dashed area shows $\widetilde{HRP}$ and $\widetilde{HRW}$, the filled area shows RP and RW. (a) Subject 18: Weak perfect substitute preferences. (b) Subject 26: Weak Nash preferences. (c) Subject 25: Weak Rawlsian preferences. (d) Subject 19: Selfish preferences.

4.3 Survey Data: Household Expenditures

We now illustrate our methods using data from the Spanish Continuous Family Expenditure Survey (Encuesta Continua de Presupuestos Familiares, abbreviated ECPF).[13] This panel is a quarterly budget survey, ranging from 1985-1997, that interviews Spanish households for up to a maximum of eight consecutive quarters on their consumption expenditures.[14] From this data, we use a sub-sample of couples with and without children, where the husband is employed full-time and the wife is outside the labor force.[15] Moreover, we exclude durable goods and focus exclusively on consumption expenditures on non-durable consumption categories.[16] Overall, the data we use contains 21,866 observations on 3,134 Spanish households.

Table 4 reports summary statistics for the calculated efficiency indices across all households. As seen from this table, homothetic efficiency is very close to utility maximisation efficiency. For example, the mean across all observations and households of Hev is 0.9960 (compared to Vev which is 1.0000). Thus, the consumption choices of the households' seem to be very well explained by homothetic preferences.

			efficiency			
Index	mean	min	1st quantile	median	3rd quantile	max
Aei	0.9998	0.9698	1.0000	1.0000	1.0000	1.0000
Hei	0.9917	0.9518	0.9890	0.9936	0.9965	1.0000
Vev	1.0000	0.9998	1.0000	1.0000	1.0000	1.0000
Hev	0.9960	0.9865	0.9936	0.9978	0.9996	1.0000

Table 4: Efficiency indices (ECPF).

Table 5 report results from the power analysis. As discussed in the Introduction, allowing for errors in revealed preference tests often leads to a loss in power, which may render the analysis practically meaningless. This concern is clearly warranted from Table 5. Specifically, standard utility maximisation (see rows 1,3 and 5) has barely any power against uniformly random behaviour, which means that Garp is unable to reject irrational consumption behaviour. On the other hand, since Harp is a stronger condition, we expect it to have more power against irrational behaviour. As seen from rows 2, 4 and 6, Harp has substantially more power against uniformly random behaviour than Garp. In fact, while the average power across households never exceeds 9 percent for Garp, the power of Harp is above 90 percent for all but a few households (for non-deflated total expenditure and expenditure deflated by Hev).

The current analysis also allow us to analyse the potential loss in power of adjusting expenditure by efficiency in revealed preference tests. Consider once again rows 2, 4 and 6 in Table 5. We see that while the power loss can be considerable when deflating expenditure by Hei (row 4) the loss is negligible when adjusting expenditure by Hev (row 6). In other words, adjusting expenditure by Hev have small effects on the power, which rather forcefully addresses the concern that adjusting expenditure by efficiency in revealed preference testing renders the analysis meaningless. Finally, Table 6 reports the conditional power results. We find that Harp has very good power against uniformly random data which satisfies Garp.

[13] See Browning and Collado (2001) and Crawford (2010) for a detailed discussion of this data set.

[14] Households are randomly rotated at a rate of 12.5 percent per quarter.

[15] We focus on this sample to minimise the effects of non-separabilities between consumption and leisure.

[16] The non-durables are aggregated into the following 15 consumption categories: (i) food and non-alcoholic drinks at home, (ii) alcohol, (iii) tobacco, (iv) energy at home, (v) services at home, (vi) non-durables at home, (vii) non-durable medicines, (viii) medical services, (ix) transportation, (x) petrol, (xi) leisure, (xi) personal services, (xii) personal non-durs, (xiii) restaurants and bars, and (xiv) travelling.

UNCONDITIONAL POWER							
Axiom	Income deflated	mean	min	1st quantile	median	3rd quantile	max
GARP	No	0.0892	0.0000	0.0000	0.0280	0.1700	0.7080
HARP	No	0.9974	0.9700	0.9980	1.0000	1.0000	1.0000
GARP	AEI	0.0859	0.0000	0.0000	0.0260	0.1640	0.6480
HARP	HEI	0.6382	0.0000	0.3760	0.7400	0.9280	1.0000
GARP	VEV	0.0877	0.0000	0.0000	0.0260	0.1680	0.6760
HARP	HEV	0.9558	0.1300	0.9480	0.9840	0.9960	1.0000

Table 5: Unconditional power for GARP and HARP (ECPF).

CONDITIONAL POWER						
Axiom	mean	min	1st quantile	median	3rd quantile	max
HARP	0.9776	0.7400	0.9800	0.9900	1.0000	1.0000

Table 6: Conditional power for HARP (ECPF).

5 CONCLUSION

Consumer choice data often violates homothetic utility maximisation. In such cases, it would be interesting to know how close the data comes to homothetic utility maximisation. For this purpose, we introduced a non-parametric approach to estimating homothetic efficiency of demand data by generalising Heufer's (2013b) method. We introduced the Homothetic Efficiency Index (HEI) and the Homothetic Efficiency Vector (HEV) in analogy to the standard Afriat Efficiency Index (AEI) and Varian's improved violation index or Varian Efficiency Vector (VEV). As the AEI, the HEI can be interpreted as a measure of wasted income under the assumption that violations of homotheticity were due to errors in decision making. As a non-parametric approach, our method does not rely on any specific form of a utility function.

Both the HEI and the HEV can be used to adjust data by deflating expenditure to reconstruct bounds on preferred and worse sets. This is motivated by a concept called e- and $\mathbf{h}$-rationalisation which is similar to a concept used recently by Halevy et al. (2012): For efficiency close to 100%, there still exists a utility function that adequately explains the data as the result of homothetic utility maximisation with minor errors.

We applied the method to two data sets. This empirical analysis illustrates how recoverability based on adjusted homothetically revealed preferred relations allows a detailed analysis of preferences at the individual level. It also demonstrates how a data set that has very low power against the alternative hypothesis of random behaviour can still be useful when testing for the stronger condition of homothetic utility maximisation. We find that efficiency can be very high for homotheticy, that tests for HARP have far greater power than tests for GARP, and that adjusting choices by efficiency measures has negligible effects on test power.

We expect that our results help in analysing experimental, survery, and field data. It will help to test the assumption of homotheticity before estimating homothetic utility functions, to quantify the extent of the violation of homotheticity, to analyse preferences in detail without the need of estimating parameters, and to increase test power for data sets which have too litle power against the alternative hypothesis of random behaviour.

The approach could easily be translated to production analysis. As homotheticity of production is assumed in many applications, a non-parametric test that provides a measure for homothetic efficiency independent of a specific production function should at the very least be a useful screening device and robustness check before parameters of a homothetic production function are estimated. Non-parametric recoverability of technological information as suggested by Varian (1984a) could also be carried out in analogy to the recoverability of preferences in our paper.

A APPENDIX

A.1 *Proofs*

A.1.1 *Proof of Theorem 4*

Proof For (1) ⇒ (2) and (3) ⇒ (1), Varian's (1983) proof can be applied with minor obvious adjustments. We will prove (2) ⇒ (3).

As in Varian (1983), define $U(\mathbf{x}) = \min_i\{U^i\mathbf{p}^i\mathbf{x}\}$. It can be easily verified that $U \in \mathcal{U}$ and that U is homothetic; what remains to be shown is that U e-rationalises Ω. Suppose Ω satisfies $\text{Harp}(e)$ and there exists $\mathbf{x}$ such that $U(\mathbf{x}) \geq U(\mathbf{x}^i)$ and $\mathbf{x}^i\ \mathrm{P}^0(e)\,\mathbf{x}$. Then $e\,\mathbf{p}^i\mathbf{x}^i = e > \mathbf{p}^i\mathbf{x}$. By continuity and monotonicity of U, there then exists $\mathbf{y} > \mathbf{x}$ such that $\mathbf{p}^i\mathbf{y} = e$ and $U(\mathbf{y}) > U(\mathbf{x})$. By $\text{Harp}(e)$, $eU^i \leq \min_j\{U^j\mathbf{p}^j\mathbf{x}^i\}$, and with $\mathbf{p}^i\mathbf{y} = e$ we obtain $\mathbf{p}^i\mathbf{y}U^i \leq \min_j\{U^j\mathbf{p}^j\mathbf{x}^i\}$. Suppose $U(\mathbf{x}^i) = U^k\mathbf{p}^k\mathbf{x}^i$; then $\mathbf{p}^i\mathbf{y}U^i \leq U^k\mathbf{p}^k\mathbf{x}^i = U(\mathbf{x}^i) \leq U(\mathbf{x})$. But $U(\mathbf{y}) = \min_j\{U^j\mathbf{p}^j\mathbf{y}\}$, so $U(\mathbf{y}) \leq U^i\mathbf{p}^i\mathbf{y}$. Then $U(\mathbf{y}) \leq U(\mathbf{x}^i) \leq U(\mathbf{x})$, but $\mathbf{y} > \mathbf{x}$, which contradicts montonicity.

Suppose instead that there exists $\mathbf{x}$ such that $U(\mathbf{x}) > U(\mathbf{x}^i)$, and $\mathbf{x}^i\ \mathrm{R}^0(e)\,\mathbf{x}$ but not $\mathbf{x}^i\ \mathrm{P}^0(e)\,\mathbf{x}$. Then $\mathbf{p}^i\mathbf{x} = e$, and we obtain $U(\mathbf{x}) \leq U(\mathbf{x}^i) \leq U(\mathbf{x})$, which implies $U(\mathbf{x}) = U(\mathbf{x}^i)$, a contradiction. Thus, U e-rationalises Ω. ∎

A.1.2 *Proof of Theorem 5*

Proof By induction. Suppose $u \in \mathcal{U}$ is homothetic and e-rationalises the data. By definition, we can assume without loss of generality that u is homogenous of degree 1. Because u is concave, we only need to consider the vertices of the closure of $\widehat{HRP}_{\hat{e}}$, that is, we only need to check if $u(\hat{t}_{i,\diamond}\mathbf{x}^i) < u(\mathbf{x}^\diamond)$ is possible.

Step 1 By homogeneity of degree 1, $u([\mathbf{p}^i\mathbf{x}^\diamond/e^2]\mathbf{x}^i) = [\mathbf{p}^i\mathbf{x}^\diamond/e^2]u(\mathbf{x}^i)$. Let $\mathbf{y} = e/[\mathbf{p}^i\mathbf{x}^\diamond]\mathbf{x}^\diamond$; then $\mathbf{p}^i\mathbf{y} = e$, and therefore $\mathbf{x}^i\ \mathrm{R}^0(e)\,\mathbf{y}$. Suppose $u(\hat{t}_{i,\diamond}\mathbf{x}^i) < u(\mathbf{x}^\diamond)$. Then $u(\mathbf{x}^i) < (e^2/[\mathbf{p}^i\mathbf{x}^\diamond])u(\mathbf{x}^\diamond)$. But with $e \leq 1$, $(e^2/[\mathbf{p}^i\mathbf{x}^\diamond])u(\mathbf{x}^\diamond) \leq (e/[\mathbf{p}^i\mathbf{x}^\diamond])u(\mathbf{x}^\diamond) = u(\mathbf{y})$. Then $u(\mathbf{x}^i) < u(\mathbf{y})$, but $\mathbf{x}^i\ \mathrm{R}^0(e)\,\mathbf{x}^\diamond$, so u cannot e-rationalise Ω. Thus, $([\mathbf{p}^i\mathbf{x}^\diamond]/e^2)u(\mathbf{x}^i) \geq u(\mathbf{x}^\diamond)$.

Step 2 Assume without loss of generality that

$$\hat{t}_{1,n} = \frac{\mathbf{p}^1\mathbf{x}^2}{e^2}\frac{\mathbf{p}^2\mathbf{x}^3}{e^2}\cdots\frac{\mathbf{p}^{n-1}\mathbf{x}^n}{e^2}$$

and that $\hat{t}_{1,\diamond} = \hat{t}_{1,n}[\mathbf{p}^n\mathbf{x}^\diamond]/e^2$. Suppose $\hat{t}_{1,n}u(\mathbf{x}^1) \geq u(\mathbf{x}^n)$. Then

$$\hat{t}_{1,\diamond}u(\mathbf{x}^1) = \hat{t}_{1,n}u(\mathbf{x}^1)\frac{\mathbf{p}^n\mathbf{x}^\diamond}{e^2} \geq u(\mathbf{x}^n)\frac{\mathbf{p}^n\mathbf{x}^\diamond}{e^2} \geq u(\mathbf{x}^0),$$

where the last inequality follows from Step 1. Thus, $\hat{t}_{1,\diamond}u(\mathbf{x}^1) \geq u(\mathbf{x}^\diamond)$.

By induction, Steps 1 and 2 show that $u(\hat{t}_{i,\diamond}\mathbf{x}^i) \geq u(\mathbf{x}^\diamond)$ for all i and all homothetic $u \in \mathcal{U}$ which e-rationalise Ω. That concludes the proof for $\widehat{HRP}$. The second subset then follows from the definition of $\widehat{HRW}_{\hat{e}}$. ■

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