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ECONOMIC PAPERS

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Efficient Formulas and Computational Efficiency for Glove Games

Ruhr Graduate School in Economics
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#456

Imprint

Ruhr Economic Papers

Published by

Ruhr-Universität Bochum (RUB), Department of Economics
Universitätsstr. 150, 44801 Bochum, Germany

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Rheinisch-Westfälisches Institut für Wirtschaftsforschung (RWI)
Hohenzollernstr. 1-3, 45128 Essen, Germany

Editors

Editorial Office

Ruhr Economic Papers #456

Responsible Editor: Wolfgang Leininger

ISSN 1864-4872 (online) – ISBN 978-3-86788-515-7

Ruhr Economic Papers #456

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Bibliografische Informationen der Deutschen Nationalbibliothek

Die Deutsche Bibliothek verzeichnet diese Publikation in der deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über: *http://dnb.d-nb.de* abrufbar.

http://dx.doi.org/10.4419/86788515
ISSN 1864-4872 (online)
ISBN 978-3-86788-515-7

Julia Belau[1]

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Abstract

A well known and simple game to model markets is the glove game where worth is produced by building matching pairs. For glove games, different concepts, like the Shapley value, the restricted Shapley value or the Owen value, yield different distributions of worth. Moreover, computational effort of these values is in general very high. This paper provides efficient allocation formulas of the component restricted Shapley value and the Owen value for glove games in case of efficient coalitions.

JEL Classification: C71

Keywords: Glove game; imbalanced market; Shapley value; Owen value; efficiency; computational complexity

December 2013

1 Julia Belau, TU Dortmund and RGS Econ – I thank Wolfgang Leininger for comments. All remaining errors are mine. Financial support by the RGS Econ is gratefully acknowledged.– All correspondence to: Julia Belau, Technische Universität Dortmund, Faculty of Economics and Social Science, Chair of Microeconomics, Vogelpothsweg 87, 44227 Dortmund, Germany, e-mail: julia.belau@tu-dortmund.de

1 Introduction

A well known and simple game from cooperative game theory is the glove game, introduced by Shapley and Shubik (1969), where a pair of gloves produces worth which has to be distributed among the agents holding the gloves. This game has a nice economic interpretation and it is used to analyze simple markets (cf. Shapley and Shubik, 1969).

Consider such a glove game with four right-glove holders $(r_1, ..., r_4)$ and two left-glove holders (l_1, l_2) where l_1, r_1 and l_2, r_2 build a matching pair while the other right-glove holders stay alone. Let a pair produce a worth of 1. Distribution of worth according to the Shapley value (Shapley, 1953), the component restricted Shapley value (Aumann and Drèze, 1974) and the Owen value (Owen, 1977) are given in Table 1.

Table 1: Payoffs for the glove game

glove holder	Shapley value	AD-value	Owen value
l_1, l_2	0.7333	0.5	0,8333
r_1, r_2	0.1333	0.5	0,1667
r_3, r_4	0.1333	0	0

Source: compare Belau (2011).

We see that the Shapley value does not distinguish wether a right-glove holder actually builds a matching pair (i.e., is productive or not). Therefore, we would like to have an allocation rule taking into account the actual matching pair, that is, the coalition structure. The component restricted Shapley value accounts for the coalition structure, but it underestimates the actual imbalancedness of the market. In case of minimal winning coalitions, it even treats imbalanced markets as balanced markets which stands into contradiction with taking into account the scarceness of the left-glove holders, or, in other words, the existence of alternatives these players would have outside their actual coalition. The Owen value both accounts for the coalitional structure and the degree of imbalancedness.

In general, computational complexity of the analyzed values is very high. Shapley and Shubik (1969) derive an efficient formula of the Shapley value for glove games. This paper provides efficient formulas of the component restricted Shapley value and the Owen value for glove games in case of efficient coalition structures, that is, minimal winning coalitions.

2 Framework: (TU)-Games and Allocation Rules and the Glove Game

Let $N = \{1, ..., n\}$ be the (nonempty and finite) playerset and $\mathbb{V}_N := \{v : 2^N \longrightarrow \mathbb{R} | v(\emptyset) = 0\}$ the set of all *characteristic functions*, that is, a function $v \in \mathbb{V}_N$ describes the underlying game and assigns to any coalition $K \subseteq N$ its worth $v(K)$. A *game with transferable utility (TU-game)* is a tupel (N, v). An *allocation rule* $Y : \mathbb{V}_N \longrightarrow \mathbb{R}^n$ distributes the worth of any TU-game among the players. One of the most popular allocation rules is the Shapley value (Shapley, 1953):

$$Sh_i(N, v) := \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!(|N| - |K| - 1)!}{|N|!} [v(K \cup \{i\}) - v(K)]$$

where $MC_i^v(K) := v(K \cup \{i\}) - v(K)$ is called the *marginal contribution* of player i for coalition $K \cup \{i\}$ in the TU-game (N, v), that is, the surplus player i creates in game v when entering coalition K. The Shapley value assigns to any player i the average marginal contribution over all orders of N.

A partition $\mathcal{P}$ of N is called *coalition structure* where $\mathcal{P}(i) \in \mathcal{P}$ denotes the coalition that contains player $i \in N$ and $\mathbb{P}_N$ denotes the set of all coalition structures of N. A *TU-game for coalition structures* is a tupel $(N, v, \mathcal{P})$ and an *allocation rule for coalition structures* is a function $Y : \mathbb{V}_N \times \mathbb{P}_N \longrightarrow \mathbb{R}^n$, distributing worth of any TU-game for coalition structres among the players. Aumann and Drèze (1974) define the component restricted Shapley value (denoted by *Aumann-Drèze value* AD) for every $(N, v, \mathcal{P})$ as follows:

$$AD_i(N, v, \mathcal{P}) := Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}).$$

Here, the whole game is restricted to the coalition of a player.

In contrast, Owen (1977) defines the Owen value:

$$Ow_i(N, v, \mathcal{P}) := |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} [v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})] \tag{1}$$

where $\Sigma(N, \mathcal{P})$ is the set of all orders σ over N that are compatible with the coalition structure $\mathcal{P}$ (i.e. $\forall\, i, j \in P \in \mathcal{P}$ we have $|\sigma(i) - \sigma(j)| < |P|$) and $K_i(\sigma)$ is the set of players that come before player i and including i under order σ.

A very popular (TU)-game to describe and analyze markets is the *glove game*, introduced by Shapley and Shubik (1969). The set of players is split into the set of left-glove holders L and the set of right-glove holders R, that is, $L \cup R = N$ and $L \cap R = \emptyset$. The worth of a coalition $K \subseteq N$ is the number of matching pairs available in K, that is, $v_{gg}(K) := \min(|R \cap K|, |L \cap K|)$.[1]

If $|R| \neq |L|$, we say that the market is *imbalanced* (and *balanced* otherwise). In an imbalanced market, players from the smaller set are called *strong players* (due to

[1] A matching pair could also create worth different to 1, but all analyzed allocation rules are additive which ensures $Y(N, x \cdot v) = x \cdot Y(N, v)$ for all $x \in \mathbb{R}$.

their scarceness) and players from the larger set *weak players*. For notational reasons, denote by $S := \min(|L|, |R|)$ the number of strong players and by $W := \max(|L|, |R|)$ the number of weak players, respectively.

3 The (efficient) Shapley value and AD-value for Glove Games

For Shapley-based values, computational effort is in general very high.

Lemma 1. *Approximated from below, computational effort of the Shapley value is at least of order $\mathcal{O}\left(n \cdot 2^{n+1}\right)$ (higher than polynomial order).*

Proof. We approximate computational effort of computing $MC_i^{v_{gg}}$ and $|K|!(|N| - |K| - 1)!$ by 1 and by computing $(|N| - 1)!$, respectively. Summation is over all subsets of $N \setminus \{i\}$, hence, $2^{|N|-1}$ subsets. Therefore, computational effort can be approximated from below by computing $2^{|N|-1}$ times the expressions $(|N|-1)!$, $|N|!$ and $MC_i^{v_{gg}}$ which has to be done for every agent $i \in N$ individually. Computational effort of $k!$ is of order $\mathcal{O}(k^2 log(k))$ (bottom-up multiplication). Hence, approximated from below, overall computational effort is at least of order

$$
\begin{aligned}
&\mathcal{O}\left(|N| \cdot 2^{|N|-1} \cdot \left((|N|-1)^2 log(|N|-1)|N|^2 log(|N|) + 1\right)\right) \\
&> \mathcal{O}\left(|N| \cdot 2^{|N|-1} \cdot \left(\underbrace{(|N|-1)^2 log^2(|N|-1)}_{>2 \text{ for } |N|\geq 3}\right)^2\right) \\
&> \mathcal{O}\left(|N| \cdot 2^{|N|-1} \cdot 2^2\right) = \mathcal{O}\left(|N| \cdot 2^{|N|+1}\right).
\end{aligned}
$$

□

For the Shapley-value, Shapley and Shubik (1969) show:

$$
Sh_i(N, v_{gg}) = \begin{cases} \frac{1}{2} + \frac{W-S}{2 \cdot S} \sum\limits_{k=1}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is strong player} \\ \frac{1}{2} - \frac{W-S}{2 \cdot W} \sum\limits_{k=0}^{S} \frac{W!S!}{(W+k)!(S-k)!} & \text{, if } i \text{ is weak player} \end{cases}. \tag{2}
$$

Here we see that the Shapley value does not distinguish wether a weak player actually builds a matching pair (i.e., is productive or unproductive).

Lemma 2. *Computational effort of the formula given by (2) is of polynomial order.*

Proof. First of all, one only needs to compute 2 expressions and not an expression for every $i \in N$ individually. The summation can be approximated from above by $S - 1$ times computing the expressions $(|N| - S)!$ and $S!$ in the nominator and again $(|N| - S)!$ and $S!$ in the denominator. Hence, computational effort can be approximated by $\mathcal{O}((|N|-S)^4 \cdot S^4 \cdot (S-1) \cdot log^2(|N|-S) \cdot log^2(S))$ (and computing the fraction in front of the sum, but this will not change the polynomial order). □

For the component restricted Shapley value, the AD-value, the allocation formula for the glove game is easily calculated, just restrict the Shapley allocation to the coalition of a player: For all $i \in N$, set $S_i := \min(|L \cap \mathcal{P}(i)|, |R \cap \mathcal{P}(i)|) \geq 0$ and $W_i := \max(|L \cap \mathcal{P}(i)|, |R \cap \mathcal{P}(i)|) \geq 1$. If $S_i = 0$ (i.e., either i stays alone as a singleton or is joined by the same type of gloves only), no matching pair exists in this coalition and the player obtains a payoff of zero. For $S_i > 0$ we have

$$AD_i(N, v_{gg}, \mathcal{P}) = \begin{cases} \frac{1}{2} + \frac{W_i - S_i}{2 \cdot S_i} \sum\limits_{k=1}^{S_i} \frac{W_i! S_i!}{(W_i+k)!(S_i-k)!} & \text{, if } i \text{ is strong player in } \mathcal{P}(i) \\ \frac{1}{2} - \frac{W_i - S_i}{2 \cdot W_i} \sum\limits_{k=0}^{S_i} \frac{W_i! S_i!}{(W_i+k)!(S_i-k)!} & \text{, if } i \text{ is weak player } \mathcal{P}(i) \end{cases}.$$

We see that only imbalancedness *within* the own coalition is taken into account. An interesting and economically important case is the case of minimal winning coalitions, that is, the coalition structure consists of matching pairs and singletons only. Consider $\mathcal{P} =$ such that $P = \{l_j, r_k\} \vee |P| = 1 \; \forall \; P \in \mathcal{P}$, then we have

$$AD_i(N, v_{gg}, \mathcal{P}) = \begin{cases} \frac{1}{2} & \text{, if } i \text{ builds a pair} \\ 0 & \text{, if } i \text{ stays alone} \end{cases}$$

We see that in this case, the AD-value splits the worth equally among the matching-pair-players, that is, as if we had a balanced market.

Remark 1. *Computational effort of the original AD-value as well as the modified version for glove games and minimal winning coalitions is negligible. However, imbalancedness of the market is underestimated/ignored.*

4 The (efficient) Owen value for Glove Games

Consider the case of minimal winning coalitions again.

Definition 1. *We call a coalition structure $\mathcal{P}$ efficient, if only minimal winning coalitions are build and no strong player stays alone: $\forall \; P \in \mathcal{P} : P \subseteq \{l_i, r_j\}$ and if for some $l_i \in L$ we have $\{l_i\} \in \mathcal{P}$, then $\nexists \; r_j \in R$ such that $\{r_j\} \in \mathcal{P}$ and if for some $r_j \in R$ we have $\{r_j\} \in \mathcal{P}$, then $\nexists \; l_i \in L$ such that $\{l_i\} \in \mathcal{P}$.*

Theorem 1. *For all efficient coalition structures $\mathcal{P}$, the Owen value for the glove game is given by*

$$Ow_i(N, v_{gg}, \mathcal{P}) = \begin{cases} 1 - \frac{(S-1)!}{2 \cdot W!} \sum\limits_{k=0}^{S-1} \frac{(W-(k+1))!}{(S-(k+1))!} & \text{, if } i \text{ is a strong player} \\ \frac{(S-1)!}{2 \cdot W!} \sum\limits_{k=0}^{S-1} \frac{(W-(k+1))!}{(S-(k+1))!} & \text{, if } i \text{ is a weak matching-pair-player} \\ 0 & \text{, if } i \text{ stays alone} \end{cases} \tag{3}$$

Here we see that both the coalition structure (matching vs. no matching) and the level of imbalancedness (S and W) is taken into account.

Proof. Due to the form of an efficient coalition structure, we have for all $P \in \mathcal{P}$: $|P| \leq 2$ and $\Sigma(N, \mathcal{P})$ only contains orders where pairs (l, r) are next to each other (lr or rl). Hence, to analyze $\Sigma(N, \mathcal{P})$, we only have to consider orders of the components of $\mathcal{P}$, having in mind that each matching-pair-component has two possibilities. Therefore, $|\Sigma(N, \mathcal{P})|$ is the number of possibilities to order the components of $\mathcal{P}$ times $2^{\#\text{ of matching pairs}}$ (each pair has two possibilities to be ordered).
Due to the form of efficient coalition structures, the number of components of $\mathcal{P}$ is equal to W and the number of matching pairs is equal to S. Hence, we have

$$|\Sigma(N, \mathcal{P})| = W!2^S.$$

Since $|P| \leq 2$, we have

$$MC_i^{v_{gg}}(\sigma) \leq 1 \, \forall \, i \in N.$$

Consider $P \in \mathcal{P}$ such that $P = \{i\}$. If there is any matching candidate before i in order σ, the pair-partner of this candidate will be before i, too. Therefore, we have $MC_i^{v_{gg}}(\sigma) = 0$ and

$$Ow_i(N, v_{gg}, \mathcal{P}) = 0 \, \forall \, i \text{ such that } \{i\} \in \mathcal{P}.$$

For any weak player i who forms a matching pair we note: matching pairs before i in order σ do not affect $MC_i^{v_{gg}}(\sigma)$ since the worth created by this pair is created independently of using $K_i(\sigma)$ or $K_i(\sigma)\backslash\{i\}$. As all strong players (= matching candidates) before i in order σ appear with their matching partner, we have $MC_i^{v_{gg}}(\sigma) = 0$ whenever i is before his matching partner in order σ. If i's matching partner is before i in order σ and there is a singleton weak player before i's matching pair, we also have $MC_i^{v_{gg}}(\sigma) = 0$, because in $K_i(\sigma)\backslash\{i\}$, i's matching partner already creates worth with this singleton weak player. Hence,

$$MC_i^{v_{gg}}(\sigma) = 1 \Leftrightarrow \begin{cases} i\text{'s matching partner is before } i \text{ in order } \sigma \text{ and there are} \\ \text{at most other matching pairs before } i\text{'s matching partner.} \end{cases}$$

This happens how many times? There can be $k = 0, ..., S-1$ matching pairs before i's matching pair in σ. For each such k we have

$$\underbrace{(S-1)}_{\text{for } 1^{\text{st}} \text{ pair}} \cdot \underbrace{(S-2)}_{\text{for } 2^{\text{nd}} \text{ pair}} \cdot \ldots \cdot \underbrace{(S-k)}_{\text{for } k^{\text{th}} \text{ pair}} \cdot \underbrace{(|\mathcal{P}| - (k+1))}_{\text{remaining pairs and singletons}} \cdot 2^{S-1}$$

possibilities, where the 2^{S-1} drops from the fact that all matching pairs but i's can occur with two orders. This can be rewritten as

$$\frac{(S-1)!}{(S-(k+1))!} \cdot (W-(k+1))! \cdot 2^{S-1}.$$

Hence, $MC_i^{v_{gg}}(\sigma) = 1$ for

$$\sum_{k=0}^{S-1}\left(\frac{(S-1)!(W-(k+1))!}{(S-(k+1))!}\cdot 2^{S-1}\right) = (S-1)!\cdot 2^{S-1}\cdot \sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!}$$

different $\sigma \in \Sigma(N, \mathcal{P})$. Therefore,

$$Ow_i(N, v_{gg}, \mathcal{P}) = \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!}$$

for all weak players i who form a matching pair.

Since the Owen value is efficient (i.e., $\sum_{i\in N} Y_i = v(N)$), we have that

$$\sum_{i\in N} Ow_i = \#\text{ of matching pairs } = S.$$

Using this and that $Ow_i(N, v_{gg}, \mathcal{P}) = 0\ \forall\ i$ such that $\{i\} \in \mathcal{P}$, we have

$$\sum_{i \text{ builds matching pair}} Ow_i = S$$

Furthermore, the Owen value assigns equal payoffs to symmetric components. All components of the form $P = \{l, r\}$ are symmetric and hence, $Ow_l + Ow_r = \frac{S}{S} = 1$ for each matching pair (l, r). Using this, we get the Owen allocation for strong players: $Ow_{\text{strong player}} = 1 - Ow_{\text{weak player in matching pair}}$. And hence, finally,

$$Ow_i(N, v_{gg}, \mathcal{P}) = \begin{cases} 1 - \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!} & \text{, if } i \text{ is a strong player} \\ \frac{(S-1)!}{2\cdot W!}\sum_{k=0}^{S-1}\frac{(W-(k+1))!}{(S-(k+1))!} & \text{, if } i \text{ is a weak matching-pair-player} \\ 0 & \text{, if } i \text{ stays alone} \end{cases}$$

□

Theorem 2. *Consider a glove game $(N, v_{gg}, \mathcal{P})$ and let $\mathcal{P}$ be efficient with $S > 0$. Then, computational complexity of (1) is at least $\mathcal{O}\left(\left\lfloor\frac{|N|}{2}\right\rfloor! \cdot |N| + |N|!\right)$ (higher than polynomial order), while computational complexity of (3) is of polynomial order.*

Proof. Consider the original formula of the Owen value (Equation (1)). Approximating computational effort of the marginal contribution by 1 for each $\sigma \in \Sigma(N, \mathcal{P})$, computational effort of the sum can be approximated from below by $|\Sigma(N, \mathcal{P})|$ computations. This has to be multiplied by $|N|$ (calculation has to be done for each agent $i \in N$). Additionally, $\Sigma(N, \mathcal{P})$ has to be computed. In case of efficient coalition structures and $S > 0$, $|\Sigma(N, \mathcal{P})|$ can be approximated from below by $\left\lfloor\frac{|N|}{2}\right\rfloor!$: there are W components in $\mathcal{P}$ and, hence, $W!$ permutations of components and for every of the S pairs, there are two inner permutations. Now neglect multiplicity due to inner permutations and use that $W \leq \frac{|N|}{2}$.

To compute $\Sigma(N, \mathcal{P})$, one has to check for each possible order σ over N wether $\sigma \in \Sigma(N, \mathcal{P})$. As $\mathcal{P}$ is efficient, this is checking wether $|\sigma(i) - \sigma(j)| = 1$ for each pair (i, j) in $\mathcal{P}$. There are $|N|!$ possible orders over N and S pairs.

Hence, computational complexity is, approximated from below, at least of order

$$\mathcal{O}\left(\left\lfloor \frac{|N|}{2} \right\rfloor ! \cdot |N| + |N|! \cdot S\right) \geq \mathcal{O}\left(\left\lfloor \frac{|N|}{2} \right\rfloor ! \cdot |N| + |N|!\right)$$

Now consider the new formula given in Equation (3). Following the arguments in the proof of Lemma 2, the expression is of polynomial order. □

5 Conclusion

This paper studies glove games: simple cooperative games that are used to analyze markets. In case of glove games, we approximate and compare computational complexity of the original Shapley formula and the modified one by Shapley and Shubik (1969). We derive modified formulas for the Owen value (and the component restricted Shapley value) in the economically important case of efficient coalition structures (minimal winning coalitions). These expressions are used to explain the drawback of the Shapley value and its component restricted pendant: either the actual coalition structure or the imbalancedness of the market is ignored. The Owen value accounts for both. Furthermore, we show that the new modified formula is computationally more efficient and the original one.

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