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## Measuring Distribution of Power of Political Agreements

technische universität dortmund
Ruhr Graduate School in Economics

#326

# Imprint

## Ruhr Economic Papers

Published by

Ruhr-Universität Bochum (RUB), Department of Economics
Universitätsstr. 150, 44801 Bochum, Germany

Technische Universität Dortmund, Department of Economic and Social Sciences
Vogelpothsweg 87, 44227 Dortmund, Germany

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Rheinisch-Westfälisches Institut für Wirtschaftsforschung (RWI)
Hohenzollernstr. 1-3, 45128 Essen, Germany

## Editors



## Editorial Office



## Ruhr Economic Papers #326

Responsible Editor: Wolfgang Leininger



ISSN 1864-4872 (online) – ISBN 978-3-86788-375-7

Ruhr Economic Papers #326

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## Bibliografische Informationen der Deutschen Nationalbibliothek

Die Deutsche Bibliothek verzeichnet diese Publikation in der deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über: *http://dnb.d-nb.de* abrufbar.

http://dx.doi.org/10.4419/86788375
ISSN 1864-4872 (online)
ISBN 978-3-86788-375-7

Julia Belau[1]

# A New Outside Option Value for Networks: The Kappa-Value – Measuring Distribution of Power of Political Agreements

## Abstract

*In an economic or social situation where agents have to group in order to achieve common goals, how can we calculate the coalitional rents of the agents arising from the coalition formation? Once we have formalized the situation via a TU-game and a network describing the economic structure, we can apply different allocation rules to assign the coalitional rents to the agents. We specifically analyze situations where parties with a specific vote distribution in a parliament have to build agreements in order to reach some required quorum. In this situation, we want to measure the (relative) distribution of power. We analyze the allocation rules called Position value (Meessen (1988) and Borm et al. (1992)) and graph-χ-value (Casajus (2009)). Applying the generalized framework (Gómez et al. (2008)), a framework where coalitions are not established yet, we find that the graph-χ-value does not differ for networks referring to the same coalition while the Position value takes into account the specific role of an agent within the network, i.e. the communication path. We define and characterize a new outside option sensitive value, the Kappa-value, which takes into account both outside options and the role of an agent within the network.*

*JEL Classification: C71, D85, H10*

*Keywords: Cooperative games; graph-restricted games; networks; position value; outside options; minimal winning coalitions*

*April 2012*

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# 1. Introduction

In an economic or social situation where agents have to group in order to achieve common goals, how can we calculate the coalitional rents of the agents arising from the coalition formation? First of all we have to formalize the situation at the ex post stage, i.e. when the coalitions are already formed. The worth produced by a coalition can be formalized appropriately by cooperative games with transferable utility (TU-games). In general, any coalition can be formed. In this paper we will analyze situations where coalition formation is restricted by communication structures, i.e. the economic or social structure is described by a network which captures the (bilateral) relations between agents and agents can only form a coalition if they are connected in such a communication network. This approach was introduced by Myerson (1977). We will use undirected networks to describe the relations between the agents while it would also be possible to use directed networks, i.e. the communication has a fixed direction. This idea was analyzed by González-Arangüena et al. (2008). Further, they use games where the worth of a coalition depends on the order in which angents enter the coalition, i.e. generalized characteristic functions as introduced by Nowak and Radzik (1994). In our application communication is not directed and the order of entry does not change the worth of the coalition.
Once we have found an appropriate TU-game and a network describing the economic or social structure, we can determine the coalitional rent of the agents via specific allocation rules.

Now we imagine a situation where the economic structure is not established yet, the ex ante stage. Modelling the situation in this generalized framework gives the opportunity to analyze allocation rules more detailed, for example for forecasting issues. Following Gómez et al. (2008), we take into account all possible structures that could arise and we use the information we have at this stage to describe the likelihood of this structures. Formally, we use the information we have to build a probability distribution over all possible networks. We will use the likelihood of a network to occur while it would also be possible to use a probability distribution over all bilateral relations between the agents. This approach assumes the independence of relations. Gómez et al. (2008) give some reasons against this assumption, for example incompatabilities: the relation between two agents can make a relation between one of them and a third agent impossible. In our application it is much more reasonable to consider the likelihood of a whole structure. That formalizes the situation at the ex ante stage. Finally, generalizing the deterministic (ex post) allocation rules, we can forecast the coalitional rent of the agents and analyze further properties of the allocation rules.

For the allocation rules, we will focus in this paper on the Position value, introduced by Meessen (1988) and further analyzed by Borm et al. (1992) and Slikker (2005) for the ex post stage and generalized by Ghintran et al. (2010) for the ex ante stage, and the graph-$\chi$-value, introduced by Casajus (2009) for the ex post stage. We will

define the graph-$\chi$-value for the ex ante stage. The Position value takes into account the role of the agent within a network while the graph-$\chi$-value takes into account the outside options an agent has (bargaining positions ect.).
We will specifically analyze political agreements, i.e. situations where parties with a specific vote distribution in a parliament have to build agreements in order to reach some required quorum (for example to pass a bill). Using ex ante allocation rules, we will measure the (relative) distribution of power of the parties. We will show that in situations like that the graph-$\chi$-value does not take into account the position of a party in the political spectrum, i.e. the specific communication path of the network, while the Position value does not take into account the bargaining position of a party that could also form agreements with another members of the parliament.
We define and characterize a new allocation rule, the Kappa-value, which can be used as a measure for distribution of power that is outside-option-sensitive and takes into account the position of an agent within the communcation path.

The paper is structured as follows: We start with a leading example for political agreements and define the determinsitic framework (ex post situation) and the probabilistic framework (ex ante situation). Following the idea of combining the properties of the two analyzed values, section 3 will give an axiomatic characterization of new solution consept. Finally, section 4 concludes.

# 2. A Leading Example and the Framework

## 2.1. Agreement Games: Leading Example

**Example 1.** *In a parliament, there are members of five parties (*$1, 2, 3, 4$ *and* $5$*). The votes are distributed corresponding to the following percentages*[1]*:*

Table 1: Percentages of parties in the parliament

| party | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| percentage | 17 | 35 | 30 | 5 | 13 |

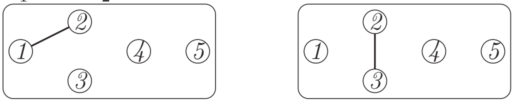
*For special types of bills to be passed, more than 50% of the votes are required.*
*The political orientation of the parties can be described as follows: Party 1 is left orientated while parties 2 and 3 are middle-left, party 4 middle right and party 5 right orientated:*

*2*
*1 4 5*
*3*

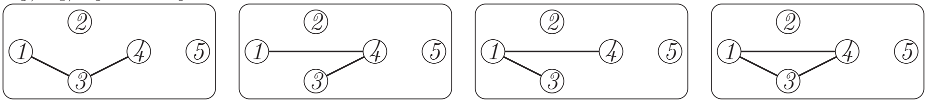
[1] Distribution based on the state election in Saarland, Germany. Further details in the example as political orientation or coalition formation are fictitious.

*Hence, there will never be an agreement between party 1 and 5 due to ideological incompatabilities. There will only be minimal agreements, i.e. once a coalition reaches more than 50%, there will be no more party entering the coalition. Parties that are not able to find partners in order to reach more than 50% will stay alone (i.e. there is nothing like an opposition).*

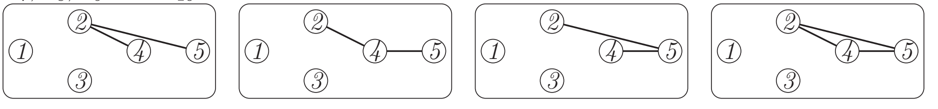
*We are left with the following networks that occur with non-zero probability:*
$L_1$ *and* $L_2$*:*



$L_3, L_4, L_5$ *and* $L_6$*:*



$L_7, L_8, L_9$ *and* $L_{10}$*:*



*Or, formally, as presented in table 2*

Table 2: Possible structures

| $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ | $L_9$ | $L_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\{12\}$ | $\{23\}$ | $\{13, 34\}$ | $\{14, 34\}$ | $\{13, 14\}$ | $\{13, 14, 34\}$ | $\{24, 25\}$ | $\{24, 45\}$ | $\{25, 45\}$ | $\{24, 25, 45\}$ |

*We are interested in the distribution of the political power of the parties. We report the political power assigned by the Position value (Meessen (1988) and Borm et al. (1992)), denoted by* $\pi$*, and the graph-*$\chi$*-value (Casajus (2009)), denoted by* $\chi^{\#}$*, to all possible networks in Table 3.*

Table 3: Ex post values

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\chi_1^\#$ | $\chi_2^\#$ | $\chi_3^\#$ | $\chi_4^\#$ | $\chi_5^\#$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_1$ | $^1/_2$ | $^1/_2$ | 0 | 0 | 0 | $^5/_{12}$ | $^7/_{12}$ | 0 | 0 | 0 |
| $L_2$ | 0 | $^1/_2$ | $^1/_2$ | 0 | 0 | 0 | $^7/_{12}$ | $^5/_{12}$ | 0 | 0 |
| $L_3$ | $^1/_4$ | 0 | $^1/_2$ | $^1/_4$ | 0 | $^7/_{18}$ | 0 | $^7/_{18}$ | $^2/_9$ | 0 |
| $L_4$ | $^1/_4$ | 0 | $^1/_4$ | $^1/_2$ | 0 | $^7/_{18}$ | 0 | $^7/_{18}$ | $^2/_9$ | 0 |
| $L_5$ | $^1/_2$ | 0 | $^1/_4$ | $^1/_4$ | 0 | $^7/_{18}$ | 0 | $^7/_{18}$ | $^2/_9$ | 0 |
| $L_6$ | $^1/_3$ | 0 | $^1/_3$ | $^1/_3$ | 0 | $^7/_{18}$ | 0 | $^7/_{18}$ | $^2/_9$ | 0 |
| $L_7$ | 0 | $^1/_2$ | 0 | $^1/_4$ | $^1/_4$ | 0 | $^5/_9$ | 0 | $^2/_9$ | $^2/_9$ |
| $L_8$ | 0 | $^1/_4$ | 0 | $^1/_2$ | $^1/_4$ | 0 | $^5/_9$ | 0 | $^2/_9$ | $^2/_9$ |
| $L_9$ | 0 | $^1/_4$ | 0 | $^1/_4$ | $^1/_2$ | 0 | $^5/_9$ | 0 | $^2/_9$ | $^2/_9$ |
| $L_{10}$ | 0 | $^1/_3$ | 0 | $^1/_3$ | $^1/_3$ | 0 | $^5/_9$ | 0 | $^2/_9$ | $^2/_9$ |

*We grouped the possible networks by the formed coalition, i.e. within a group the structure only differs by the path the parties are connected to each other. We note that the graph-$\chi$-value does not differ within a group, while the Position value takes into account the specific position of a party in the network. On the other hand, the Position value does not take into account the outside options of the parties, i.e. the distribution of the votes.*

## 2.2. Deterministic framework: The ex post situation

A game in characteristic function form (or TU game) $(N, v)$ consists of a non-empty and finite player set $N = \{1, ..., n\}$ and coalition function (or characteristic function) $v \in \mathbb{V}(N) := \{v : 2^N \longmapsto \mathbb{R} | v(\emptyset) = 0\}$, assigning to every coalition $K \subseteq N$ the worth of the coalition. A game is called zero-normalized if $v(\{i\}) = 0 \; \forall \; i \in N$. Note that any $v \in \mathbb{V}(N)$ can be zero-normalized. A payoff vector $\varphi(N, v)$, assigning a payoff to every player $i \in N$, is called solution concept.

A very popular solution concept for (TU) games is the Shapley value (Shapley (1953)):

$$Sh_i(N, v) := \sum_{K \subseteq N \setminus \{i\}} \frac{k!(n-1-k)!}{n!} \left[v(K \cup \{i\}) - v(K)\right],$$
$$\text{where } k = |K|, n = |N|.$$

The Shapley value assigns to every player its share of his marginal contributions $[v(K \cup \{i\}) - v(K)]$.

A network (or graph) $L$ is a set of links $ij$, connecting the nodes/players $i, j \in N$. Let $L^c := \{ij | i, j \in N\}$ be the network in which all players are connected to each other, the complete network. We denote the set of all networks by $\mathcal{L} := \{L | L \subseteq L^c\}$. We call a pair $(N, L)$ cooperation or communication structure (CO) and a triple $(N, v, L)$ is called a CO-game (or cooperation/communication situation). A CO-value (or solution concept for a cooperation situations) is a payoff vector $\varphi(N, v, L)$, assigning a payoff to every player $i \in N$.

We say that players $i$ and $j$ are connected in the network $L$ if there exists a path $ii_1, i_1i_2, ..., i_kj \in L$, $i_1, ..., i_k \in N$. Connected players form components in a cooperation structure $(N, L)$ and these components build a partition on the player set $N$. We denote this partition by $\mathcal{C}(N, L)$; $C_i := C_i(N, L) \in \mathcal{C}(N, L)$ is the component of all players connected with player $i \in N$.

Meessen (1988) and Borm et al. (1992) assign to every cooperation structure an alternative (TU) game, the so called link-game (or arc game) in which the links in the network are the players:

$$v^N(L) := v^L(N)$$

where $v^L(K) := \sum\limits_{S \in \mathcal{C}(K, L|_K)} v(S)$ is the graph-restricted game introduced by Myerson (1977).
Using this, he introduces the Position value $\pi$:

$$\pi_i(N, v, L) := \sum_{\lambda \in L_i} \frac{1}{2} Sh_\lambda(L, v^N),$$

where $L_i$ = set of links including player $i$.

The Position value takes into account the role of links in which a player is (directly) involved. For example: Let $N = \{1, 2, 3\}$ and let $v(K) = 1$ if $K = N$ and $v(K) = 0$ otherwise. Consider the following structure:
①—②—③
The links are $\lambda_1 := \{12\}$ and $\lambda_2 := \{23\}$. Both links are needed to create worth and $\lambda_1$ is needed as much as $\lambda_2$, i.e. they are symmetric in the link-game. Hence, they both obtain half of the worth: $Sh_{\lambda_1} = Sh_{\lambda_2} = \frac{1}{2}$. The Position value assigns to every player half of the payoff of every link he is involved in, i.e. $\pi_1 = \frac{1}{2}Sh_{\lambda_1} = \frac{1}{4}$, $\pi_2 = \frac{1}{2}Sh_{\lambda_1} + \frac{1}{2}Sh_{\lambda_2} = \frac{1}{2}$ and $\pi_3 = \frac{1}{2}Sh_{\lambda_2} = \frac{1}{4}$. The interpretation is that player 2 is "needed more" to create the worth because he connects the three players. His position in the network is stronger which comes through the fact that he is involved in more links than the other two players. Hence, he obtains a higher payoff.

(Casajus (2009)) introduces an outside option sensitive CO-value. To reflect all (productive) outside options, he defines for every network $L$ the corresponding lower outside-option graph (LOOG)(For detailed motivation see Casajus (2009), page 5.): $L(i, N) := L|_{C_i} \cup \{jk \in L^c | j \in C_i, k \in N \setminus C_i\}$. For notational reasons, if the player set is fixed, we will only write $L(i)$. Using this, he defines the the graph-$\chi$-value:

$$\chi_i^{\#}(N, v, L) := Sh_i(N, v^{L(i)}) + \frac{v(C_i) - Sh_{C_i}(N, v^{L(i)})}{|C_i|},$$
$$\text{where } \varphi_K = \sum_{i \in K} \varphi_i$$

The graph-$\chi$-value values outside options by adding/substracting some share of the Shapley values of its component to the Shapley value of the player.

## 2.3. Probabilistic framework: The ex ante situation

We follow the definition of the generalized framework of Gómez et al. (2008). A probability distribution $p$ over $\mathcal{L}$ is a function $p : \mathcal{L} \longmapsto [0, 1]$ such that $\sum_{L \in \mathcal{L}} p(L) = 1$. Let $\mathbb{P}(\mathcal{L})$ denote the set of all probability distributions on $\mathcal{L}$. We call $(N, v, p)$ a probabilistic cooperation situation (pCO). A pCO-value is a payoff vector $\varphi(N, v, p)$, assigning a payoff to every player $i \in N$.

Ghintran et al. (2010) define the probabilistic Position value via a probabilistic extension of the coalition function but it turns out that this definition is equivalent to the following (see Ghintran et al. (2010), page 12):

$$\pi(N, v, p) = \sum_{L \in \mathcal{L}} p(L) \pi(N, v, L)$$

Ghintran et al. (2010) find that the probabilistic Position value can be characterized directly via probabilistic pendants of the characterizing axioms of the determinsitc Position value.

Belau (2011) defines a generalized framework for coalition situations (coalitions without inner link structure) and defines probabilistic values as the probabilistic $\chi$-value for coalition situations. Following this and the way Gómez et al. (2008) and Ghintran et al. (2010) define pCO-values, we define the probabilistic graph-$\chi$-value as follows:

**Definition 1.** *For any $(N, v, p)$ the probabilistic graph-$\chi$-value is given by*

$$\chi^{\#}(N, v, p) := \sum_{L \in \mathcal{L}} p(L) \chi^{\#}(N, v, L)$$

# 3. A new Outside Option Value: The Kappa-value

In this section we will define a new solution concept for CO-games which takes into account both outside options and the role of an agent within the network. In the leading example, we have seen that the graph-$\chi$-value does not differ within the class of networks referring to the same coalition, but it takes into account outside options. While the latter is not true for the Position value, it takes into account the role of an agent within a network. Hence, we will further analyze the characterizing axioms of the two values in order to combine those axioms leading to outside option sensitivity and the importance of the role of an agent.

Slikker (2005) shows that the Position value is characterized by the two following axioms:

**Definition 2** (Component Efficiency (CE))**.** *For all* $C \in \mathcal{C}(N, L)$ *we have* $\sum_{i \in C} \varphi_i(N, v, L) = v(C)$.

We interpret the components as productive unions. A productive union should get the payoff it creates.

**Definition 3** (Balanced Link Contributions (BLC))**.** *For all* $i, j \in N$, $i \neq j$ *and* $v \in \mathbb{V}_0$ *we have*

$$\sum_{\lambda \in L_j} \left[\varphi_i(N, v, L) - \varphi_i(N, v, L - \lambda)\right] = \sum_{\lambda \in L_i} \left[\varphi_j(N, v, L) - \varphi_j(N, v, L - \lambda)\right].$$

Slikker (2005) argues that BLC "deals with the loss players can inflict on each other. The total threat of a player towards another player is defined as the sum over all links of the first player of the payoff differences the second player experiences if such a link is broken." BLC states that the total threat of a player towards another player should be equal to the reverse total threat.

We note that the Position value also satisfies Component Decomposability:

**Definition 4** (Component Decomposability (CD))**.** *Fir all* $i \in C \in \mathcal{C}(N, L)$ *we have*
$\varphi_i(N, v, L) = \varphi_i(C, v|_C, L|_C)$.

> Component Decomposability states that the player's outside world does not affect payoffs within a component. Neither the potential coalitions between players in the component and outside the component, nor the coalition structure. Hence, it stands in contradiction to outside-option-sensitivity.[2]

Hence, if we want a value to account for outside options but still to take into account the role of a player within the network, we need to weaken BLC and fill the gap with an outside option axiom.

Note that a connected graph lacks outside options. This brings us to the following weaker version of BLC:

[2]Belau (2011), page 17

**Definition 5** (Weak Balanced Link Contributions (WBLC))**.** *For all connected $L$ and all $i, j \in N$ and $v \in \mathbb{V}_0$ we have*

$$\sum_{\lambda \in L_j} \left[\varphi_i(N, v, L) - \varphi_i(C_i(N, L - \lambda), v|_{C_i(N, L-\lambda)}, L|_{C_i(N, L-\lambda)})\right]$$
$$= \sum_{\lambda \in L_i} \left[\varphi_j(N, v, L) - \varphi_j(C_j(N, L - \lambda), v|_{C_j(N, L-\lambda)}, L|_{C_j(N, L-\lambda)})\right].$$

**Lemma 1.** *If $\varphi$ satisfies CE and WBLC, it coincides with the Position value for all connected graphs.*

Find the proof in the appendix.

If we combine Lemma 1 with the presence of CD, we will have a characterization of the Position value. Hence, we need to weaken CD. We use the axiom that accounts for outside options given by Casajus (2009):

**Definition 6** (Outside Option Consistency (OO))**.** *For all $i, j \in C \in \mathcal{C}(N, L)$ we have*

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i(N, v, L(i)) - \varphi_j(N, v, L(j)).$$

**Theorem 1.** *There is a unique CO-value that satisfies CE, OO and WBLC.*

*Proof.* We follow the idea of the uniquenessproof of the graph-$\chi$-value (Casajus (2009)). Let $\varphi$ satisfy CE, OO and WBLC. For $i, j \in C_i$ we have $L(i) = L(j)$. By OO:

$$\varphi_i(N, v, L) - \varphi_i(N, v, L(i)) = \varphi_j(N, v, L) - \varphi_j(N, v, L(i))$$

Summing up over $j \in C_i$ and using CE gives

$$|C_i| \left[\varphi_i(N, v, L) - \varphi_i(N, v, L(i))\right] = v(C_i) - \varphi_{C_i}(N, v, L(i))$$

Since $L(i)$ is connected, Lemma 1 implies $\varphi_i(N, v, L(i)) = \pi(N, v, L(i))$ for all $i \in N$ and hence we have

$$\varphi_i(N, v, L) = \pi_i(N, v, L(i)) + \frac{v(C_i) - \pi_{C_i}(N, v, L(i))}{|C_i|} \tag{1}$$

which uniquely determines $\varphi$.
It is easily shown that the value given by equation (1) satisfies CE and OO. WBLC follows after some calculations (see appendix). □

We call the value given by (1) "Kappa-value" and denote it by $\kappa$. Note that it differs from the graph-$\chi$-value by using the Position value instead of the Myerson value (Myerson (1977)). Note that the Myerson value also does not differ within the class of networks referring to the same coalition.

For the leading example, the distribution of power assigned by the Kappa-value can be found in Table 4:

Table 4: Ex post Kappa-values

| | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_4$ | $\kappa_5$ |
|---|---|---|---|---|---|
| $L_1$ | 0.477 | 0.523 | 0 | 0 | 0 |
| $L_2$ | 0 | 0.523 | 0.477 | 0 | 0 |
| $L_3$ | 0.350 | 0 | 0.378 | 0.272 | 0 |
| $L_4$ | 0.333 | 0 | 0.333 | 0.333 | 0 |
| $L_5$ | 0.378 | 0 | 0.350 | 0.272 | 0 |
| $L_6$ | 0.3525 | 0 | 0.3525 | 0.295 | 0 |
| $L_7$ | 0 | 0.417 | 0 | 0.292 | 0.292 |
| $L_8$ | 0 | 0.408 | 0 | 0.333 | 0.258 |
| $L_9$ | 0 | 0.408 | 0 | 0.258 | 0.333 |
| $L_{10}$ | 0 | 0.410 | 0 | 0.295 | 0.295 |

# 4. Conclusion and further Research

In order to measure the distribution of power for political agreements, we analyzed the Position value, which takes into account the position of a player in the network, and the graph-$\chi$-value, which takes into account outside options. For political agreement situations we found that the graph-$\chi$-value does not differ within eqivalence classes of networks referring to the same coalition. This motivated the idea to find a value that captures the fact that the actual network, i.e. the communication path, matters since the payoff of an agent depends on the position within the network and still accounts for outside options. We gave an axiomatic characterization of a new solution concept which combines the named properties, the Kappa-value.

In order to forecast the distribution of power at a stage where coalitions (or networks) are not established yet, Belau (2011) defines a generalized framework, the ex ante framework, for coalition situations (coalitions without inner link structure) and defines snd characterizes the ex ante $\chi$-value for coalition situations. It is pointed out that a direct characterization via ex ante versions of the ex post axioms is not sufficient if outside options are taken into account. Gómez et al. (2008) define the ex ante framework for communication situations. Ghintran et al. (2010) define the ex ante Position value and find that it can be characterized directly via ex ante pendants of the characterizing axioms of the ex post Position value. It would be interesting to analyze an ex ante version of the Kappa-value and to find a characterization.

# A. Proofs

### Proof of Lemma 1

*Proof.* We follow the idea of the proof for the Myerson value of Casajus (2009). Since the position value can be characterized by CE and BLC, it satisfies WBLC since it follows from BLC together with CD. Let $\varphi$ and $\psi$ satisfy CE and WBLC. Suppose $N$ is the minimal player set such that $\varphi$ and $\psi$ differ on a connected graph. We must have $|N| > 1$, because for $|N| = 1$ we would have a contradiction by CE due to the connectedness of the graph. Suppose that $L$ is the minimal connected graph on N such that $\varphi \neq \psi$.
Let $i, j \in N$. By WBLC and the minimality of $L$ we have

$$\begin{aligned}
|L_j|\varphi_i(N,v,L) - |L_i|\varphi_j(N,v,L) &= \sum_{\lambda\in L_j} \varphi_i(N,v,L-\lambda) - \sum_{\lambda\in L_i} \varphi_j(N,v,L-\lambda) \\
&= \sum_{\lambda\in L_j} \psi_i(N,v,L-\lambda) - \sum_{\lambda\in L_i} \psi_j(N,v,L-\lambda) \\
&= |L_j|\psi_i(N,v,L) - |L_i|\psi_j(N,v,L) \\
\Leftrightarrow |L_j|\left[\varphi_i(N,v,L) - \psi_i(N,v,L)\right] &= |L_i|\left[\varphi_j(N,v,L) - \psi_j(N,v,L)\right]
\end{aligned}$$

Summing up over $j \in C_i = N$ (connected graph), we have by CE:

$$\underbrace{\left(\sum_{j\in N} |L_j|\right) \frac{|N|}{|L_i|}}_{>0} \left[\varphi_i(N,v,L) - \psi_i(N,v,L)\right] = v(N) - v(N) = 0$$

and hence $\varphi_i(N,v,L) = \psi_i(N,v,L)$ □

### Proof of WBLC in Theorem 1

*Proof.* To see WBLC, first note that the Position value satisfies WBLC by satisfying BLC and CD. Let $L$ be connected and $i, j \in N$, then we have $C_i = C_j = N$ and $L(i) = L(j) = L$. Hence,

$$\begin{aligned}
&\sum_{\lambda\in L_j} \varphi_i(N,v,L) - \sum_{\lambda\in L_j} \varphi_j(N,v,L) \\
&= \sum_{\lambda\in L_j} \left[\pi_i(N,v,L) + \underbrace{\frac{v(N) - \pi_N(N,v,L)}{|N|}}_{=0 \text{ by CE}}\right] - \sum_{\lambda\in L_i} \left[\pi_j(N,v,L) + \underbrace{\frac{v(N) - \pi_N(N,v,L)}{|N|}}_{=0 \text{ by CE}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda \in L_j} \pi_i(N, v, L) - \sum_{\lambda \in L_i} \pi_j(N, v, L) \\
&\overset{WBLC}{=} \sum_{\lambda \in L_j} \pi_i(C_i(N, L-\lambda), v|_{C_i(N,L-\lambda)}, L|_{C_i(N,L-\lambda)}) \\
&\quad - \sum_{\lambda \in L_j} \pi_j(C_j(N, L-\lambda), v|_{C_j(N,L-\lambda)}, L|_{C_j(N,L-\lambda)}) \\
&= \sum_{\lambda \in L_j} \varphi_i(C_i(N, L-\lambda), v|_{C_i(N,L-\lambda)}, L|_{C_i(N,L-\lambda)}) \\
&\quad - \sum_{\lambda \in L_j} \varphi_j(C_j(N, L-\lambda), v|_{C_j(N,L-\lambda)}, L|_{C_j(N,L-\lambda)})
\end{aligned}
$$

where the last step drops from the fact that $L|_{C_k(N,L-\lambda)}$ is connected on $C_k(N, L-\lambda)$ and by CE. $\square$

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