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Continuous Extensions and Recoverability

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Jan Heufer[1]

Revealed Preference and Nonparametric Analysis – Continuous Extensions and Recoverability

Abstract

This paper shows how revealed preference relations, observed under general budget sets, can be extended using closure operators which impose certain assumptions on preferences. Common extensions are based on the assumption that preferences are convex and/or monotonic, but we also consider satiated single-peaked preferences. For the obtained extended relations, the paper provides necessary and sufficient conditions for the existence of continuous complete extensions of the revealed preference relation. These results lead to a nonparametric analysis of revealed preference data which allows to recover all that can be said about a decision maker's preferences. The approach makes explicit what additional assumptions are imposed on the revealed preference relation. For example, Varian's (1982) "revealed preferred set" imposes monotonicity and convexity. The approach focuses strictly on what is observable. In particular, it does not assume that we observe all bundles on a budget among which the decision maker is indifferent.

JEL Classification: C12, C14, C91, D00, D11, D12, D81, G11

Keywords: Consistency of binary relations; continuous extension; decision theory; demand theory; GARP; nonparametric analysis; revealed preference

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1 Introduction

1.1 *Overview*

This paper shows how revealed preference relations, observed under fairly general budget sets, can be extended using closure operators which impose certain assumptions on preferences. For the obtained extended relations, the paper provides necessary and sufficient conditions for the existence of continuous complete extensions of the revealed preference relation. These results naturally lead to a nonparametric analysis of revealed preference data. The approach focuses strictly on what is observable. In particular, it does not assume that the researchers observes all bundles on a budget among which the decision maker (DM) is indifferent. That is, a single element of a budget is observed as a choice, even if the demand function is multi-valued.

This approach has several advantages. It allows to distinguish between different assumptions on preferences, and when recovering preferences it highlights, at every step, which assumption in particular is imposed on the revealed preference relation. The primitives of the model are strictly restricted to what is observable by a researcher, and the paper is explicit about the differences between conclusions backed by directly observed revealed preference and conclusions based on reasoning that goes beyond revealed preference. The framework is sufficiently general to allow extensions of revealed preference relations other than the usual extensions such as monotonicity, convexity, or homotheticity. It further highlights the problems which arise when we do not observe the entire demand function; for example, with general budget sets, it is difficult to test for non-satiation or single-peakedness of preferences without assuming convexity. Furthermore, as it provides the tools needed to test assumptions on preferences when observing choices on general budget sets, it allows to conduct economic experiments which can recover more about a subject's preference than what can be achieved with induced competitive budgets.

1.2 *Point of Departure and Motivation*

Suppose we observe a finite number of choices of a consumer from competitive budget sets. If these choices satisfy a condition called "cyclic consistency" by Afriat (1967), then (and only then) there exists a continuous, monotonic, and concave utility function which is maximised by these choices, taking the corresponding budget as a constraint. Furthermore, Afriat was able to construct such a utility function using the so called Afriat numbers. Varian (1982, 1983) introduced a condition—the "Generalised Axiom of Revealed Preference" (Garp)—which is equivalent to cyclic consistency and particularly easy to test.

This result, now commonly known as Afriat's Theorem, is remarkable for many reasons. It assumes competitive budget sets which occur quite naturally in a competitive economy; real data is therefore very likely to be generated by choices on such budgets. This, and the ease with which Garp can be tested, highlights the operational aspects of revealed preference theory. Afriat's Theorem, as shown by Varian (1982), also states that if the choices could have been generated by a continuous and non-satiated utility function, then they could have also been generated by a monotonic and concave utility function. But is this a "good" or "bad" aspect of revealed preference theory when choices are made on competitive budgets? It can be considered to be bad in the sense that it shows the limitations of dealing with data consisting of choices from competitive budgets: Choices generated by a non-satiated utility function are empirically indistinguishable from (or observationally equivalent to) choices generated by a monotonic and concave utility function. In terms of Chambers et al. (2010), the theory of non-satiated utility maximisation is the falsifiable closure of the theory of monotonic concave utility maximisation.

Putting the falsifiability aspect aside, the data we observe can be put to very good use. Varian (1982) shows how to construct sets of revealed preferred and revealed worse bundles, based on consistency with Garp, which are boundaries on a consumer's indifference set. Thus, based on the assumption that a consumer's preference is indeed monotonic and convex he shows how to recover everything that can be set about a consumer's preference without recovering anything that cannot be said. In particular, his analysis is completely nonparametric and does not rely on parameter estimates of a particular functional form of a utility function which represents a consumer's preference. His definition of revealed preferred and worse sets relies on Garp for competitive budget sets. This makes it difficult to apply these sets to the framework of, for example, Yatchew (1985) or Forges and Minelli (2009) (see below). The results in this paper make the recovery of such sets straightforward.

Given that competitive budgets are a natural starting point in economics, extensive examinations of revealed preference and the nonparametric analysis based on it are well motivated. In the last years, revealed preference analysis has received a lot of attention by experimental economists (see, for example, Sippel 1997, Mattei 2000, Harbaugh and Krause 2000, Harbaugh et al. 2001, Andreoni and Miller 2002, Février and Visser 2004, Chen et al. 2006, Fisman et al. 2007, Choi et al. 2007a, Banerjee and Murphy 2007, Dickinson 2009, Dawes et al. 2011). Economic experiments, if appropriately incentivised, are an almost ideal setting to test if subjects' preferences satisfy certain assumptions, as it allows to collect the kind of data we need without many caveats.

Most of these experiments rely on experimentally induced competitive budget sets. This may be partly because such budgets are easy to describe to subjects, and partly because the revealed preference literature mainly provides nonparametric techniques for such budgets. However, the graphical representation of budgets to subjects and the possibility for subjects to select the desired bundle with a computer mouse introduced in Fisman et al. (2007) and Choi et al. (2007a) (see also Choi et al. 2007b) should make it quite possible to provide subjects with more general budget sets. The analysis in this paper provides some of the tools needed to analyse such data.

Why would it be interesting to present subjects more general budget sets? First, it would allow to test if preferences really are monotonic and convex. For example, several experiments have examined subjects' social preferences; but social preferences can contain spite or inequality aversion, and social preferences may therefore not be monotonic. Second, suppose a subject has Leontief-type preferences. Varian's (1982) analysis shows that recovering large parts of the "revealed worse" set of this subject would require very steep budget sets. Very steep budget sets, however, would result in very expensive experiments if there are also many subjects with perfect substitute preferences. More general budget sets would allow to recover large parts of a subject's preference without having to worry about excessive costs.

1.3 *Related Literature*

Afriat's (1967) work emphasised the operational aspects of revealed preference theory based on competitive budgets. Varian (1982, 1983) showed how to use such revealed preference relations for nonparametric analysis and how to test several assumptions of the particular forms of utility. Yatchew (1985) extended the approach for budget sets which are finite unions of convex sets. Matzkin and Richter (1991) showed how to test revealed preference data for the existence of a rationalising strictly concave of utility function. Matzkin (1991) and Forges and Minelli (2009) provided revealed preference axioms and an Afriat's Theorem for more general budget sets. Heufer (2010) provided an analysis of choices on probability simplices, Kalandrakis (2010) provided an analysis of binary choices of voters. These two papers dispense with the monotonicity assumption and analyse the data for consistency with a certain point of satiation. Demuynck (2009) analysed closure operators on partial (incomplete) relations and provided conditions for the existence of complete extensions which satisfy certain assumptions.

1.4 *Outline*

Section 2 introduces the basic notation and the concepts used in the paper. In particular, it defined several specific binary relations, introduced the basic assumptions on budgets, decision makers, and observables, defined closure operators, and defines the Generalised Axiom of Revealed Preferences (Garp) as the main point of departure. Section 3 defines our concept of rationalising extensions and shows that Garp guarantees the existence of a complete, continuous, and rationalising extension of the revealed preference relation. The section further discusses the problems of verifying non-satiation of preferences and shows how to test for single-peakedness of preferences. It then goes on to provide several new axioms which are necessary and sufficient for the existence of convex, monotonic, or single-peaked rationalising extensions and for some combinations of the characteristics. Section 4 shows how large parts of a preference underlying observed choices can be recovered when imposing additional assumptions, and it discusses computational aspects of the framework. Section 5 discusses several other assumptions which can be imposed on revealed preference relations. Section 6 concludes.

2 PRELIMINARIES

2.1 *General Definitions*

We use the following notation: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers excluding 0. For all $x, y \in \mathbb{R}^L$, $L \in \mathbb{N}$, $L \geq 2$, we denote $x \geqq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, L$; $x \geq y$ if $x \geqq y$ and $x \neq y$; $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, L$. We denote $\mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_i \geqq \mathbf{0}\}$ and $\mathbb{R}^L_{++} = \{x \in \mathbb{R}^L : x > \mathbf{0}\}$ where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^L$.

Let $\mathbb{X} \subseteq \mathbb{R}^L_+$ be a convex set. If $\mathbb{X} \subset \mathbb{R}^L_+$, then $\mathbb{X}$ is also compact. Let $2^{\mathbb{X}}$ be the power set of $\mathbb{X}$. Let $N_\varepsilon(x) = \{y \in \mathbb{X} : d(x, y) < \varepsilon\}$ be the open epsilon neighbourhood of x in $\mathbb{X}$, where $d : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}_+$ is the Euclidean distance function.

For any set $S \subseteq \mathbb{X}$, the interior of S, denoted int$S$, is the set of all points $x \in S$ for which there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subset S$. The closure of S, denoted cl$S$, is the set of all $x \in \mathbb{X}$ such that for all $\varepsilon > 0$, $N_\varepsilon(x) \cap S \neq \varnothing$. The boundary of a set S is $\partial S = \text{cl}S \setminus \text{int}S$.

The *convex hull* of a set $Y \subset \mathbb{X}$ is the smallest convex set in $\mathbb{X}$ that contains Y; more precisely, it is the intersection of all convex sets in $\mathbb{X}$ containing Y. For a finite set $\{s^1, \ldots, s^n\} = S \subset \mathbb{X}$, the convex hull is simply

$$CH(S) = \left\{ x \in \mathbb{X} : \exists \lambda \in [0,1]^n, \sum_{i=1}^{n} \lambda_i = 1, x = \sum_{i=1}^{n} \lambda_i s^i \right\}. \tag{1}$$

The *monotonic hull* is

$$MH(S) = \{x \in \mathbb{X} : \exists s \in S, x \geq s\}, \tag{2}$$

and the *convex monotonic hull* is

$$CMH(S) = CH[MH(S)]. \tag{3}$$

An *affine subspace* of $\mathbb{R}^L$ is a translate of a linear subspace, that is, a subset $A = x + \mathcal{L}$, $x \in \mathbb{R}^L$ and $\mathcal{L}$ is a linear subspace of $\mathbb{R}^L$. For a set $H \subset \mathbb{R}^L$, the intersection of all affine subspaces containing H is the *affine hull* of H. The *relative interior* of a set $S \in \mathbb{R}^L$ is the interior of S within the affine hull of S. A *convex polytope* (or simply *polytope*) S in $\mathbb{R}^L$ is a set which is the convex hull $CH(Y)$ of a non-empty finite set Y of elements in $\mathbb{R}^L$. A point $y^i \in S$ is an *extreme point* of S if $y^i = \lambda\, y^j + (1-\lambda) y^k$, $\lambda \in (0,1)$, y^j, $y^k \in S$ imply that $y^i = y^j = y^k$. A convex polytope is the convex hull of all its extreme points (Brøndsted 1983, Theorem 5.10). A set $F \subset S$ is a *proper face* (or simply *face*) of S if there exists a supporting hyperplane H of S such that $F = S \cap H$, $F \neq \varnothing$, and $F \neq S$.

The dimension of a face F is the dimension of the affine hull of F. A face F is called a k−face if the dimension of F is k. A 0−face is called a *vertex*, and a 1−face is called an *edge*. A $(k-1)$−face is also called a *facet*. Then an equivalent definition of an extreme point is to say that $y \in S$ is an extreme point if $\{y\}$ is a vertex.

For two sets $S_1, S_2 \in \mathbb{R}^L$, the *Minkowski sum* is the set obtained by adding all elements of S_2 to all elements of S_1, that is,

$$S_1 + S_2 = \{x \in \mathbb{R}^L : \exists s \in S_1, \exists s' \in S_2, x = s + s'\}. \tag{4}$$

For $\lambda \in [0,1]$, the *weighted Minkowski average* is defined as

$$(1-\lambda)S_1 + \lambda\, S_2 = \{x \in \mathbb{R}^L : \exists s \in S_1, \exists s' \in S_2, x = (1-\lambda)s + \lambda\, s'\}. \tag{5}$$

Note that if S_1 and S_2 are convex, the Minkowski sum and the weighted Minkowski average of the two sets are convex as well. Further note that if $S_1 \subset S_2$, then $(1-\lambda)S_1 + \lambda\, S_2 \subset S_2$ for $\lambda \in [0,1)$.

2.2 *Binary Relations*

A binary relation Q on $\mathbb{X}$ is a set of ordered pairs of elements of $\mathbb{X}$. Let $\mathcal{Q}$ be the set of all binary relations on $\mathbb{X}$. A binary relation Q is

- *transitive* if whenever $\{(x,y),(y,z)\} \subseteq \mathrm{Q}$ then $(x,z) \in \mathrm{Q}$;
- *complete* if for all $x, y \in \mathbb{X}$, either $(x,y) \in \mathrm{Q}$ or $(y,x) \in \mathrm{Q}$;
- *continuous* if for all $x \in \mathbb{X}$ the sets $\{y \in \mathbb{X} : (x,y) \in \mathrm{Q}\}$ and $\{y \in \mathbb{X} : (y,x) \in \mathrm{Q}\}$ are closed;
- *non-satiated* if for all $x \in \mathbb{X}$ and all $\varepsilon > 0$, there exists a $y \in N_\varepsilon(x)$ such that $(y,x) \in \mathrm{Q}$ and $(x,y) \notin \mathrm{Q}$;
- *single-peaked at* ϕ if there exists one and only one $\phi \in \mathbb{X}$ such that $(\phi, x) \in \mathrm{Q}$ and $(x,\phi) \notin \mathrm{Q}$ for all $x \neq \phi$;
- *convex* if for all $x, y, z \in \mathbb{X}$ with $\{(x,z),(y,z)\} \subseteq \mathrm{Q}$, $([1-\mu]x + \mu\, y, z) \in \mathrm{Q}$ for all $\mu \in [0,1]$;
- *monotonic* if for all $x, y \in \mathbb{X}$ with $x \geq y$, $(x,y) \in \mathrm{Q}$.

For a binary relation $\mathrm{Q} \in \mathcal{Q}$, Q^{-1} denotes the *inverse*, that is, $\mathrm{Q}^{-1} = \{(x,y) \in \mathbb{X} \times \mathbb{X} : (y,x) \in \mathrm{Q}\}$. Let $\mathrm{S}(\mathrm{Q}) = \mathrm{Q} \cap \mathrm{Q}^{-1}$ denote the *symmetric* part of Q, and $\mathrm{A}(\mathrm{Q}) = \mathrm{Q} \backslash \mathrm{S}(Q)$ the *asymmetric* part of Q. For binary relations which involve symbols like $\gtrsim$ and $\succ$, we will also write $\gtrsim^{-1} = \lesssim$ and $\succ^{-1} = \prec$.

Let $\mathcal{L} : \mathcal{Q} \times \mathbb{X} \to 2^{\mathbb{X}}$ denote the set of elements $y \in X$ such that $(x,y) \in \mathrm{Q}$, that is, $\mathcal{L}(\mathrm{Q}, x) = \{y \in \mathbb{X} : (x,y) \in \mathrm{Q}\}$. Reversely, let $\mathcal{U} : \mathcal{Q} \times \mathbb{X} \to 2^{\mathbb{X}}$ be defined as $\mathcal{U}(\mathrm{Q}, x) = \{y \in \mathbb{X} : (y,x) \in \mathrm{Q}\}$.

2.3 *Budget Sets and Decision Makers*

A *budget set* or simply *budget* is a subset of $\mathbb{X}$. The only restriction we place on a budget set $B \subset \mathbb{X}$ is that is compact, that its interior is non-empty, and that for all $x \in B$ there exists a $\varepsilon > 0$ such that the interior of $N_\varepsilon(x) \cap B$ is non-empty[1]. Let $\mathcal{B}$ be the family of all such budgets.

A particular kind of budgets are the budgets faced by a competitive consumer in demand theory. These budgets are defined by a price vector $\tilde{p} \in \mathbb{R}^L_{++}$ and wealth $\tilde{w} \in \mathbb{R}_{++}$, such that $B(\tilde{p}, \tilde{w}) = \{x \in \mathbb{X} : \tilde{p} \cdot x \leq \tilde{w}\}$, where $\cdot$ denotes the dot product. Assuming homogeneity of demand, we can normalise price vectors and wealth such that $p = \tilde{p}/\tilde{w}$ and $w = 1$; we will therefore always use competitive budgets of the form $B(p,1) = B(p) = \{x \in \mathbb{X} : p \cdot x \leq 1\}$. Let $\mathcal{B}_D$ denote the family of all such budgets.

We assume that a *decision maker* (DM) can be represented by a transitive, complete, and continuous binary relation on $\mathbb{X}$. This binary relation $\succsim \in \mathcal{Q}$ represents his *preference* according to which he decides which element or elements to choose from a budget. The interpretation is as usual, i.e. $(x, y) \in \succsim$ means that to the DM x is at least as good as y. The asymmetric part $\succ = \mathrm{A}(\succsim^*)$, is interpreted as a strict preference, that is, $(x, y) \in \succ$ means that to the DM x is strictly better than y.

We will focus on particular subsets of preferences. The basic assumptions on preferences are expressed in the definitions of the following sets:

$$\succsim_T = \{\mathrm{Q} \in \mathcal{Q} : \mathrm{Q} \text{ is transitive, complete, and continuous}\}, \tag{6}$$

$$\succsim_C = \succsim_T \cap \{\mathrm{Q} \in \mathcal{Q} : \mathrm{Q} \text{ is convex}\}, \tag{7}$$

$$\succsim_M = \succsim_T \cap \{\mathrm{Q} \in \mathcal{Q} : \mathrm{Q} \text{ is monotonic}\}, \tag{8}$$

$$\succsim_{S(\phi)} = \succsim_T \cap \{\mathrm{Q} \in \mathcal{Q} : \mathrm{Q} \text{ is single-peaked at } \phi\}, \tag{9}$$

$$\succsim_{NS} = \succsim_T \cap \{\mathrm{Q} \in \mathcal{Q} : \mathrm{Q} \text{ is non-satiated}\}. \tag{10}$$

The assumption that the DM is *rational* states that when asked to choose a element from a budget set B, he will choose an x such that $(x, y) \in \succsim$ for all $y \in B$.

2.4 *From Data to Revealed Preference*

The primitives of our model, the *observations* on a DM, are a finite number of budget sets faced by the DM and the choices he makes from these sets. An observation consists of a budget $B \subset \mathbb{X}$ and a *choice* (or *demand*) $D(B) \in \mathbb{X}$ consisting of a single element, where $D : 2^{\mathbb{X}} \to \mathbb{X}$ is the *choice* or *demand function*. Let $\mathcal{M} = \{1, \ldots, M\}$, $M \in \mathbb{N}$; a set of observations is given by $\Omega = \{(x^i, B^i)\}_{i \in \mathcal{M}} \subset \mathbb{X} \times \mathcal{B}$, with $x^i = D(B^i)$, where $\{s^i\}_{i \in I}$ is shorthand notation for $\bigcup_{i \in I}\{s^i\}$. In a slight abuse of notation, we also allow $M = 0$, meaning that we have no observations on that DM. We denote $\Omega_1 = \{x^i\}_{i \in \mathcal{M}}$ and $\Omega_2 = \{B^i\}_{i \in \mathcal{M}}$.

The basic idea of revealed preference is that the choice of a DM reveals something about his preference. Because he is observed to choose $x^i \in B^i$ instead of some other $x \in B^i$, $x \neq x^i$, we conclude that he prefers x^i to x. Note that when we use the term "prefers" we mean that to the DM, x^i is at least as good as any other available alternative; in particular, x^i is preferred to itself. When we mean that to the DM, x^i is better (and not just at least as good) than some other x, we will say that he strictly prefers x^i to x.

[1] The last condition assures that even if B consists of two more more disjoint sets, all components of B have non-empty interiors

We use the observations to draw the conclusion that the DM prefers $x^i = D(B^i)$ over all $x \in B^i$. Thus, we define the revealed preference relation $\succsim^*$ as

$$\succsim^* = \{(x^i, y) \in \Omega_1 \times \mathbb{X} : y \in B^i\} \tag{11}$$

We could also draw the conclusion that the DM strictly prefers $x^i \in B^i$ over all $x \in \text{int}B^i$. This conclusion makes immediate sense if preferences are assumed to be non-satiated. If, however, there is a point of satiation $\phi \in \mathbb{X}$, and $\phi \in \text{int}B$, then $D(B) = \phi$ and ϕ would be strictly preferred to itself; but for now, we define the strict revealed preference relation $\succ^*$ as

$$\succ^* = \{(x^i, y) \in \Omega_1 \times \mathbb{X} : y \in \text{int}B^i\}. \tag{12}$$

The basic idea is then simply that $\succsim^* \subset \succsim$ and $\succ^* \subset \succ$, that is, that $\succsim$ is an extension of $\succsim^*$ (see Section 2.6). Note that $(x, y) \in \succsim^*$ if and only if $x = D(B^i)$ and $y \in B^i$ for some $i \in \mathcal{M}$, and $(x, y) \in \succ^*$ if and only if $x = D(B^i)$ and $y \in \text{int}B^i$ for some $i \in \mathcal{M}$.

Alternatively, it is often assumed that one observes multi-valued choices, i.e. a subset $D(B^i) \subseteq B^i$ of B^i with possibly more than one element. Then $\succ^*$ can be defined as $(x, y) \in \succ^*$ if $x \in D(B^i)$ and $y \in B^i \backslash D(B^i)$. Such multi-valued choices may be observable under ideal conditions; however, it is more realistic that one observes a single element as the choice from a budget.

Let

$$\Psi(\mathrm{Q}, x) = \{x^i \in \Omega_1 : (x^i, x) \in \mathrm{Q}\}; \tag{13}$$

for example, $\Psi[C_T(\succsim^*), x]$ is the set of all observed choices which are directly or indirectly revealed preferred to x.

2.5 *Closure Operators*

A *closure operator* on a binary relation $\mathrm{Q} \in \mathcal{Q}$ is a function $C : \mathcal{Q} \to \mathcal{Q}$ which is extensive, increasing, and idempotent: Q is

$$\begin{aligned} &\text{extensive if} && \mathrm{Q} \subseteq C(\mathrm{Q}), && (14)\\ &\text{increasing if} && \mathrm{Q} \subseteq \mathrm{Q}' \Rightarrow C(\mathrm{Q}) \subseteq C(\mathrm{Q}'), && (15)\\ &\text{idempotent if} && C(C(\mathrm{Q})) = C(\mathrm{Q}). && (16)\end{aligned}$$

We will apply closure operators to revealed preference relations to "impose" certain additional assumptions.

When we apply two or more closure operators to a binary relation Q, we denote this as $C_{a,b,\ldots,c}(\mathrm{Q}) = C_a(C_b(\ldots(C_c(\mathrm{Q}))\ldots))$.

The revealed preference relation does not account for the fact that every element of $\mathbb{X}$ is as good as itself. We therefore need a very basic operator, the *reflexive closure*:

$$C_R(\mathrm{Q}) = \mathrm{Q} \cup \bigcup_{x \in \mathbb{X}} \{(x, x)\}. \tag{17}$$

The second very important operator we need is the *transitive closure*. Let $C_{\tilde{T}(0)}(\mathrm{Q}) = C_R(\mathrm{Q})$ and for $i \in \mathbb{N}$,

$$C_{\tilde{T}(i)}(\mathrm{Q}) = C_{\tilde{T}(i-1)}(\mathrm{Q}) \cup \{(x, y) \in \mathbb{X} \times \mathbb{X} : \exists z \in \mathbb{X}, \{(x, z), (z, y)\} \subseteq C_{\tilde{T}(i-1)}(\mathrm{Q})\}; \tag{18}$$

then define the transitive closure as

$$C_T(\mathrm{Q}) = \bigcup_{i \in \mathbb{N}} C_{\tilde{T}(i)}(\mathrm{Q}). \tag{19}$$

We will call $C_T(\succsim^*)$ the indirect revealed preference relation. The *monotonic closure* is defined as

$$C_M(\mathrm{Q}) = C_T(\mathrm{Q} \cup \geq). \tag{20}$$

For the *convex closure*, we first define for some set $S \subset \mathbb{X}$,

$$\Upsilon(S, \mathrm{Q}, y) = CH\big[\{y\} \cup \bigcup_{x \in S} \mathcal{U}(\mathrm{Q}, x)\big], \tag{21}$$

$$\Upsilon'(S, \mathrm{Q}, y) = \Upsilon(S, \mathrm{Q}, y) \backslash CH\big(\big[\bigcup_{x \in S} \mathcal{U}(\mathrm{Q}, x)\big] \backslash \{y\}\big), \tag{22}$$

$$\Upsilon^*(S, \mathrm{Q}, y) = \mathrm{int}\Upsilon(S, \mathrm{Q}, y) \cup \Upsilon'(S, \mathrm{Q}, y), \tag{23}$$

and then

$$CE(S, \mathrm{Q}, y) = \begin{cases} 1 & \text{if } \forall x \in S, \exists z \in \mathbb{X}, z \neq x, z \in \Upsilon^*(S, \mathrm{Q}, y) \cap \mathcal{L}(\mathrm{Q}, x) \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Let $\Xi = \{x \in \mathbb{X} : \mathcal{L}(C_R(\mathrm{Q}), x) \neq \{x\}\}$, and let $|\Xi|$ denote the cardinality of Ξ. For $i \in \mathbb{N}$, let 2^{Ξ}_i be the set of all subsets of Ξ with exactly i distinct elements. Let $C_{\tilde{C}(0)}(\mathrm{Q}) = C_T(\mathrm{Q})$ and for $i \in \mathbb{N}$,

$$C_{\tilde{C}(i)}(\mathrm{Q}) = C_T\Big(C_{\tilde{C}(i-1)}(\mathrm{Q}) \cup \bigcup_{S \in 2^{\Xi}_i} \{(x, y) \in S \times \mathbb{X} : CE(S, C_T(\mathrm{Q}), y) = 1\}\Big); \tag{25}$$

then define

$$C_{\tilde{C}}(\mathrm{Q}) = \bigcup_{i \in \mathbb{N}} C_{\tilde{C}(i)}(\mathrm{Q}); \tag{26}$$

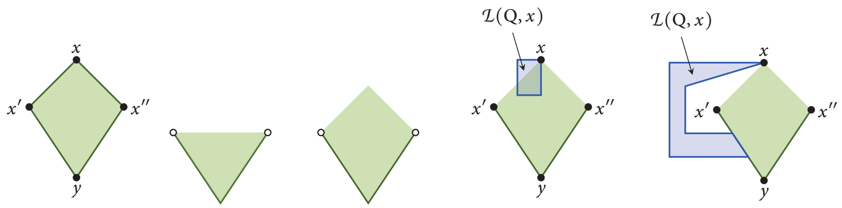
then define the convex closure as

$$C_C(\mathrm{Q}) = C_T(\{(x, y) \in \mathbb{X} \times \mathbb{X} : x \in CH(\mathcal{U}[C_{\tilde{C}}(\mathrm{Q}), y])\}). \tag{27}$$

The definition of the convex closure may require some explanation. For ease of illustration, suppose Q is an incomplete preference relation, and assume it is a subset of a complete, continuous, and convex preference R. The set $\Upsilon^*(\{x\}, \mathrm{Q}, y)$ is simply the line segment connecting x and y, excluding the point x. If (y, x) were an element of R, then by convexity $\Upsilon^*(\{x\}, \mathrm{Q}, y)$ would have to be a subset of the preferred set of y, i.e. $\Upsilon^*(\{x\}, \mathrm{Q}, y) \subseteq \mathcal{U}(\mathrm{R}, y)$. But $\mathcal{L}(\mathrm{Q}, x) \subseteq \mathcal{L}(\mathrm{R}, x)$, and if $CE(\{x\}, \mathrm{Q}, y) = 1$, the worse set of x would intersect the preferred set of y. Thus either $(y, x) \notin \mathrm{R}$ or $\{(x, y), (y, x)\} \in \mathrm{R}$; in any case, by completeness of R, we need $(x, y) \in \mathrm{R}$.

Consider now a set $S = \{x, x'\} \subseteq \Xi$, with $CE(\{x\}, Q, y) = 0$, $CE(\{x, x'\}, Q, y) = 1$, $\mathcal{U}(Q, x) = \{x\}$, and $\mathcal{U}(Q, x') = \{x'\}$. By completeness of R, we have either $(x, x') \in R$ or $(x', x) \in R$. Suppose $(x, x') \in R$. If $(y, x) \in R$, then by transitivity $(y, x') \in R$ and therefore by convexity $CH(\{x, x', y\}) \subseteq \mathcal{U}(R, x')$. But then for some $z \in \mathcal{L}(Q, x')$, either (i) $z \in \text{int}\Upsilon(S, Q, y)$ or (ii) $z \in CH(\{x', y\})$ or (iii) $z \in CH(\{x, y\})$. Case (i) is obviously a contradiction. Case (ii) implies $(z, y) \in R$ and thus $(x', y) \in R$, and with $(x, x') \in R$, we have $(x, y) \in R$. Case (iii) implies with $(x, z) \in R$ that $(x, y) \in R$. So suppose instead that $(x', x) \in R$. If we had $(y, x) \in R$, then by convexity $\Upsilon(S, Q, y) \subseteq \mathcal{U}(R, x)$. Then we can repeat the previous steps for some $z \in \mathcal{L}(Q, x)$ to obtain that $(x, y) \in R$.

When considering a set S with at least three elements (in two dimensions), the construction of Υ^* becomes relevant. Suppose that we want to check if $\{(y, x), (y, x'), (y, x'')\} \in R$ is possible, with $S = \{x, x', x''\}$ and $L = 2$. Suppose we first try this based on the assumption that $\{(x', x), (x'', x)\} \in R$ (if Q does not already contradict this possibility). Then $\Upsilon(S, Q, y)$ would be a subset of the preferred set of x. What the construction of Υ^* now does is (in the two dimensional case) to remove the edges (which in this case are facets) of the convex polygon $\Upsilon(S, Q, y)$ which do not contain the vertex $\{y\}$.[2] If the worse set of x according to Q, $\mathcal{L}(Q, x)$, intersects $\Upsilon(S, Q, y) \setminus \Upsilon^*(S, Q, y)$ (i.e., it is tangent to one of the two edges which are removed in Υ^*), this is not a contradiction, as there still is the possibility that x, x', and x'' are all indifferent according to R and y is preferred to all of them. Only if $\mathcal{L}(Q, x)$ intersects either (i) the interior of $\Upsilon(S, Q, y)$ or (ii) a facet which contains y do we have a contradiction. In case (i), this follows from continuity, and in case (ii) this follows from the same arguments as in the preceding paragraph. See Figure 1 for an illustration.



(a) The set Υ. (b) The set Υ'. (c) The set Υ^*. (d) Case (i). (e) Case (ii).

Figure 1: Illustration of the sets Υ, Υ', and Υ^*, used in the definition of the convex closure, and the two cases in which $(x, y) \in R$ if $\{(x', x), (x'', x)\} \in R$. Note that in case (i), there is already a violation of convexity even if y is not considered.

When dealing with a finite number of observations, the definitions of the transitive and convex closure are perfectly operational, that is, we can compute these closures in a finite number of steps, as the first proposition shows.

[2] In general, the construction of Υ^* removes facets of $\Upsilon(S, Q, y)$, where a "facet" is defined in terms of the dimension of $\Upsilon(S, Q, y)$, that is, in terms of the dimension of the affine hull of $\Upsilon(S, Q, y)$. For example, if in the three dimensional case $\Upsilon(S, Q, y)$ is contained in a two dimensional hyperplane, then the construction of $\Upsilon^*(S, Q, y)$ still removes edges of $\Upsilon(S, Q, y)$. If however $\Upsilon(S, Q, y)$ is also of dimension three, the construction removes two dimensional facets.

Proposition 1 *For all Ω with $M \geq 1$ observations,*

$$C_T(\succsim^*) = \bigcup_{i=1}^{M-1} C_{\tilde{T}(i)}(\succsim^*) = C_{\tilde{T}(M-1)}(\succsim^*);$$

$$C_{\tilde{C}}(\succsim^*) = \bigcup_{i=1}^{M} C_{\tilde{C}(i)}(\succsim^*) = C_{\tilde{C}(M)}(\succsim^*).$$

Proof We first show that $C_T(\succsim^*) = C_{\tilde{T}(M-1)}(\succsim^*)$. If $(x,y) \in C_R(\succsim^*)$ and $x \neq y$, then $x = x^i \in \Omega_1$. Then $(x^i, y) \in C_{\tilde{T}(1)}(\succsim^*) \backslash C_R(\succsim^*)$ implies $\{(x^i, x^j), (x^j, y)\} \subseteq C_R(\succsim^*)$ with $j \in \mathcal{M}\backslash\{i\}$, and $(x^i, y) \in C_{\tilde{T}(2)}(\succsim^*) \backslash C_{\tilde{T}(1)}(\succsim^*)$ implies $\{(x^i, x^k), (x^k, y)\} \subseteq C_{\tilde{T}(1)}(\succsim^*)$ with $k \in \mathcal{M}\backslash\{i,j\}$. Continuing in this fashion shows that $(x^i, y) \in C_{\tilde{T}(M-1)}(\succsim^*) \backslash C_{\tilde{T}(M-2)}(\succsim^*)$ implies $\{(x^i, x^\ell), (x^\ell, y)\} \subseteq C_{\tilde{T}(M-2)}(\succsim^*)$ with $\{\ell\} = \mathcal{M}\backslash\{i,j,k,\ldots\}$. Thus $(x^i, y) \in C_{\tilde{T}(M)}(\succsim^*) \backslash C_{\tilde{T}(M-1)}(\succsim^*)$ is impossible as it would require an observation with index in $\mathcal{M}\backslash\{i,j,k,\ldots,\ell\} = \varnothing$, and if $C_{\tilde{T}(i)}(\succsim^*) = C_{\tilde{T}(i+1)}(\succsim^*)$, then $C_{\tilde{T}(m)}(\succsim^*) = C_{\tilde{T}(i)}(\succsim^*)$ for all m with $M-1 \geq m \geq i$.

For $C_{\tilde{C}}(\succsim^*) = C_{\tilde{C}(M)}(\succsim^*)$, it suffices to note that $\Xi \subseteq \Omega_1$. Then $(x,y) \in C_{\tilde{C}}(\succsim^*)$ only if $x \in \Omega_1$, and $CH(\mathcal{U}[C_{\tilde{C}}(\succsim^*), y])$ is the convex hull of a (possibly empty) subset of Ω_1. ∎

The following facts are mostly straightforward.

Fact 1

1. *C_R is a closure operator.*
2. *C_T is a closure operator and $C_T(\mathrm{Q}) \in \succsim_T$.*
3. *C_M is a closure operator and $C_M(\mathrm{Q}) \in \succsim_M$.*
4. *C_C is a closure operator and $C_C(\mathrm{Q}) \in \succsim_C$.*
5. *$C_{S(\phi)}$ is a closure operator and $C_{S(\phi)} \in \succsim_{S(\phi)}$.*
6. *$C_{C,M}$ is a closure operator and $C_{C,M}(\mathrm{Q}) \in (\succsim_C \cap \succsim_M)$.*
7. *$C_{C,S(\phi)}$ is a closure operator and $C_{C,S(\phi)}(\mathrm{Q}) \in (\succsim_C \cap \succsim_{S(\phi)})$.*

Proof

1. This is obvious.
2. See Demuynck (2009) for a stronger result.
3. Is is straightforward that C_M is extensive and increasing. For idempotence, it is sufficient that $C_T(\mathrm{Q} \cup \geq) \cup \geq = C_T(\mathrm{Q} \cup \geq)$. Suppose $(x,y) \in C_T(\mathrm{Q} \cup \geq) \cup \geq$ but $(x,y) \notin C_T(\mathrm{Q} \cup \geq)$. Then $(x,y) \in \geq$, but then $(x,y) \in C_T(\mathrm{Q} \cup \geq)$, a contradiction. Monotonicity of $C_M(\mathrm{Q})$ is obvious.
4. Again it is straightforward that C_C is extensive and increasing. For idempotence, let $\Xi_C = \{x \in \mathbb{X} : \mathcal{L}(C_C(\mathrm{Q}), x) \neq \{x\}\}$ to avoid confusion. Note that $C_{\tilde{C}}$ is idempotent. Thus, if $(y,z) \in C_{\tilde{C}}[C_C(\mathrm{Q})] \backslash C_C(\mathrm{Q})$, then $y \in \Xi_C \backslash \Xi$ and therefore $y \in CH(\mathcal{U}[C_{\tilde{C}}(\mathrm{Q}), x'])$ for some $x' \notin \Xi$. For $T = \{y, y', y'', \ldots\} \subseteq \Xi_C$, $CE[T, C_C(\mathrm{Q}), z] = 1$ implies that $\Upsilon^*[T, C_C(\mathrm{Q}), z]$ contains elements of $\mathcal{L}[C_C(\mathrm{Q}), y]$. But for some $S \subseteq \Xi$, $\Upsilon^*(S, \mathrm{Q}, y) \subseteq \Upsilon^*[T, C_C(\mathrm{Q}), z]$, and $y \in \Upsilon^*(S, \mathrm{Q}, y) \cap \Upsilon^*[T, C_C(\mathrm{Q}), z]$. But then $CE[T, C_T(\mathrm{Q}), z] = 1$ and $y \in CH[\mathcal{U}(C_{\tilde{C}}, z)]$, which contradicts $(y,z) \in C_{\tilde{C}}[C_C(\mathrm{Q})] \backslash C_C(\mathrm{Q})$.
   For convexity, suppose $\{(x,z), (y,z)\} \subseteq \mathrm{Q}$. Then $\{x,y\} \in \mathcal{U}[C_C(\mathrm{Q}), z]$ and thus $\mu(x,y) \in CH(\mathcal{U}[C_C(\mathrm{Q}), z])$. Then $(\mu(x,y), z) \in C_C(\mathrm{Q})$.
5. This is obvious.
6. This follows from 3 and 4.

7. This follows from 4 and 5.

■

2.6 *Complete Extensions of Binary Relations*

A binary relation $\mathrm{Q} \in \mathcal{Q}$ is *consistent* if $C_T(\mathrm{Q}) \cap \mathrm{A}(\mathrm{Q}^{-1}) = \varnothing$. The binary relation Q' is an *extension* of Q (or Q' *extends* Q, or Q *is extended by* Q') if $\mathrm{Q} \subseteq \mathrm{Q}'$ and $\mathrm{A}(\mathrm{Q}) \subseteq \mathrm{A}(\mathrm{Q}')$; this is written $\mathrm{Q}' \overset{e}{\to} \mathrm{Q}$. Suzumura (1976) shows that a binary relation has an extension if and only if it is consistent. Demuynck (2009) proves a more general result: If a closure operator C satisfies certain restrictions, then Q has a complete extension Q' such that $\mathrm{Q}' = C(\mathrm{Q}')$ if and only if $C(\mathrm{Q}) \cap \mathrm{Q}^{-1} = \varnothing$. This is quite a remarkable result, as it allows to test certain hypothesis on the preferences of a DM.

In particular, let $\mathrm{Ext}(\mathrm{Q}, C) = \{\mathrm{Q} \in \mathcal{Q} : C(\mathrm{Q}) \overset{e}{\to} \mathrm{Q}\}$ be the set of all binary relations on $\mathbb{X}$ such that the closure operator C extends Q, and consider the following conditions:

COND 1: For every well-ordered chain $\mathrm{Q}_0 \subset \mathrm{Q}_1 \subset \ldots \subset \mathrm{Q}_n \subset \ldots$ of binary relations in $\mathrm{Ext}(\mathrm{Q}, C)$, $\bigcup_{i=0}^{n} \mathrm{Q}_i \subseteq \mathrm{Ext}(\mathrm{Q}, C)$.

COND 2: For every $\mathrm{Q} \in \mathrm{Ext}(\mathrm{Q}, C)$ such that $\mathrm{N}(\mathrm{Q}) \neq \varnothing$ (i.e., Q is not complete), there exists a non-empty $\mathrm{Q}' \subseteq \mathrm{N}(\mathrm{Q})$ such that $\mathrm{Q} \cup \mathrm{Q}' \in \mathrm{Ext}(\mathrm{Q}, C)$.

Then Demuynck (2009) shows the following:[3]

Theorem 1 (Demuynck 2009) *Suppose a closure operator C satisfies* COND *1 and 2. Then there exists a complete extension Q' of Q such that $\mathrm{Q}' = C(\mathrm{Q}')$ if and only if $C(\mathrm{Q}) \cap \mathrm{A}(\mathrm{Q}^{-1}) = \varnothing$.*

2.7 *The World According to* GARP

In the context of demand theory with consumption choices from competitive budgets, Varian (1982) introduces a condition called the *Generalised Axiom of Revealed Preference* (GARP), which is equivalent to Afriat's (1967) cyclic consistency condition. We will use GARP as the primary point of departure in our analysis.

Axiom 1 *A set of observations Ω with associated revealed preference relations $(\succsim^*, \succ^*)$ satisfies the* Generalised Axiom of Revealed Preference *(GARP) if $\Psi[C_C(\succsim^*), x^i] \cap \mathrm{int}B^i = \varnothing$ for all $i \in \mathcal{M}$.*

Remark 1 *Equivalently, Ω satisfies GARP if whenever $(x^j, x^i) \in \succ^*$ then $(x^i, x^j) \notin C_T(\succsim^*)$ for all $\{i, j\} \subseteq \mathcal{M}$. The definition above was chosen because it highlights the similarities with axioms defined below.*

By definition of $\succsim^*$, $(x, y) \in \succsim^*$ if and only if $x = x^i (= D(B^i))$ and $y \in B^i$ for some $i \in \mathcal{M}$. Then it follows from the definition of C_T that $(x, y) \in C_T(\succsim^*)$ if and only if $x = x^i$ for some $i \in \mathcal{M}$ and $y \in B^j$ for some $j \in \mathcal{M}$ with $(x^i, x^j) \in C_T(\succsim^*)$. Similarly, $(x, y) \in \prec^*$ if and only if $y = x^i$ and $x \in \mathrm{int}B^i$ for some $i \in \mathcal{M}$.

Fact 2 *For all $\mathrm{Q} \subseteq \mathcal{Q}$, $C_T(\mathrm{Q})$ is consistent.*

---

[3]Demuynck's (2009) third condition is idempotence (Eq 16).

Fact 2 is trivial.

Fact 3 *The following conditions are equivalent:*
(i) Ω *satisfies* GARP.
(ii) $C_T(\succsim^*) \cap \prec^* = \varnothing$.
(iii) $\succ^* \subseteq \mathrm{A}[C_T(\succsim^*)]$.
(iv) $\succ^* \subseteq \mathrm{A}[C_T(\succsim^*)]$ *and* $C_T(\succsim^*)$ *is consistent.*

Proof

(i) ⇒ (ii) We show ¬(ii) ⇒ ¬(i). Suppose $(x, y) \in C_T(\succsim^*) \cap \prec$. Then $x = x^i$ and $y = x^j$ for some $\{i, j\} \in \mathcal{M}$. Thus $(x^i, x^j) \in C_T(\succsim^*)$ and $(x^j, x^i) \in \succ^*$, which violates GARP.

(ii) ⇒ (i) This is obvious.

(i) ⇒ (iii) We show ¬(iii) ⇒ ¬(i). If $(x, y) \in \succ^*$, then $(x, y) \in \succsim^*$. If $(x, y) \notin \mathrm{A}(C_T(\succsim^*))$ then either $(x, y) \notin C_T(\succsim^*)$—which is already excluded—or $(y, x) \in C_T(\succsim^*)$. Thus, $(y, x) \in C_T(\succsim^*)$ and $(x, y) \in \succ^*$, which violates GARP.

(iii) ⇒ (i) We show ¬(i) ⇒ ¬(iii). If $(x, y) \in C_T(\succsim^*)$, then $(y, x) \notin \mathrm{A}(C_T(\succsim^*))$. So if $(x, y) \in C_T(\succsim^*)$ and $(y, x) \in \succ^*$, then $\succ^* \nsubseteq \mathrm{A}(C_T(\succsim^*))$.

(iii) ⇔ (iv) This is obvious (see also Fact 2).

■

From Fact 3 and Theorem 1 is follows that GARP is equivalent to the existence of a complete and transitive binary relation Q on $\mathbb{X}$ which extends $C_T(\succsim^*)$ and obeys the strict revealed preference relation $\succ^*$ as defined above.

3 CONTINUOUS EXTENSION RESULTS

3.1 *Continuous Extensions of Binary Relations*

When preferences are defined on topological spaces it is interesting to consider continuous extensions. In the context of this paper, we have assumed that $\mathbb{X} \subseteq \mathbb{R}^L_+$, hence $\mathbb{X}$ is a separable topological space. As we have assumed that the DM is representable by a continuous, complete, and transitive preference, we need to ask under what circumstances $C_T(\succsim^*)$ has a continuous extension that obeys $\succ^*$.

A utility function $u : \mathbb{X} \to \mathbb{R}$ *rationalises* a set of observations Ω if $u(x) \geq u(y)$ whenever $(x, y) \in \succsim^*$ and $u(x) > u(y)$ whenever $(x, y) \in \succ^*$. Similarly, a binary relation $\mathrm{Q} \in \succsim_T$ *rationalises* a set of observations if $(x, y) \in \mathrm{Q}$ whenever $(x, y) \in C_T(\succsim^*)$ and $(x, y) \in \mathrm{A}(\mathrm{Q})$ whenever $(x, y) \in \succ^*$. That is, Q rationalises Ω if $\mathrm{Q} \overset{e}{\to} C_T(\succsim^*)$ and $\mathrm{A}(\mathrm{Q}) \supset \succ^*$. By Theorem 1 and Fact 2, a complete and transitive $\mathrm{Q} \in \mathcal{Q}$ $\mathrm{Q} \overset{e}{\to} C_T(\succsim^*)$ always exists, so $\mathrm{A}(\mathrm{Q}) \supset \succ^*$ is the part that has potential empirical meaning.

Fact 4 *For all* $x \in \mathbb{X}$,

$$\mathcal{L}(\mathrm{A}[C_T(\succsim^*)], x) = \bigcup_{\{i \in \mathcal{M} : (x, x^i) \in C_T(\succsim^*)\}} B^i \setminus \bigcup_{\{j \in \mathcal{M} : (x^j, x) \in C_T(\succsim^*)\}} \{x^j\}.$$

We omit the proof, as it follows directly from the definition of $\mathrm{A}[C_T(\succsim^*)]$. Fact 4 simply states that the set of all elements to which x is directly or indirectly revealed preferred is (i) empty if x was not observed as a choice, (ii) is the union of all the budgets associated with choices to which x is preferred without x itself and without the choices which are preferred to x.

Lemma 1 *Suppose Ω satisfies GARP. Then for all $x \in \mathbb{X}$, there exists an open set $S \subset \mathbb{X}$ such that for all $s \in S$, $(s, x) \notin C_T(\succsim^*)$ and $\mathcal{L}(\mathrm{A}[C_T(\succsim^*)], x) \subset S$.*

Proof By Fact 4, if $x \notin \Omega_1$, then $\mathcal{L}(\mathrm{A}[C_T(\succsim^*)], x) = \varnothing$, which is why we only need to consider $x \in \Omega_1$.

GARP implies that if $(x^j, x^i) \in C_T(\succsim^*)$ and $x^j \in B^i$, then $x^j \in \partial B^i$ for all $i, j \in \mathcal{M}$; this also means that $x^i \in \partial B^i$ for all $i \in \mathcal{M}$. If $\mathcal{L}(\mathrm{A}[C_T(\succsim^*), x^i) = B^i \setminus \{x^i, x^j, \ldots, x^k\}$ with $\{i, j, \ldots, k\} \subseteq \mathcal{M}$, then we must have $B^i = B^j = \ldots, B^k$ and $x^i, x^j \ldots, x^k$ are all in ∂B^i.

Let $\mathcal{M}'(x^i) = \{j \in \mathcal{M} : (x^j, x^i) \in C_T(\succsim^*) \wedge x^j \in \partial B^j\}$. All $B \subset \mathbb{X}$ are closed by definition. For all $x \in \partial B^i$, there exists a closed neighbourhood $\mathrm{cl}N_\varepsilon(x)$ for small enough ε such that for all x^j with $x^j \notin \partial B^i$ and $(x^j, x) \in C_T(\succsim^*)$, we have $x^j \notin \mathrm{cl}N_\varepsilon(x)$. Let $\mathcal{E}$ be the set of all $\varepsilon > 0$ for which this condition holds.

Let $T(\varepsilon) = B^i \cup \bigcup_{x \in \partial B^i} \mathrm{cl}N_\varepsilon(x)$ for all $\varepsilon \in \mathcal{E}$; $T(\varepsilon)$ is closed. Let $T'(x^j, \varepsilon, \delta) = \partial T(\varepsilon) \cap \mathrm{cl}N_\delta(x^j)$ for all $\delta \in \mathcal{E}$ and for all $x^j \in \partial B^i$. Let $W(x^j, \varepsilon, \delta) = \mathrm{cl}CH(T'(x^j, \varepsilon, \delta) \cup \{x^j\})$. Choose $\bar{\varepsilon}, \bar{\delta} \in \mathcal{E}$ with $\bar{\delta} > \bar{\varepsilon} > 0$ so small that $W(x^j, \bar{\varepsilon}, \bar{\delta}) \cap \partial B^i = x^j$. Then $T(\bar{\varepsilon}) \setminus [\mathrm{int}T'(x^j, \bar{\varepsilon}, \bar{\delta}) \cup \mathrm{int}W(x^j, \bar{\varepsilon}, \bar{\delta})]$ is a closed set. See Figure 2 for an illustration.

Set $S_i' = T(\bar{\varepsilon}) \setminus \bigcup_{\{j \in \mathcal{M}'(x^i)\}} [\mathrm{int}T'(x^j, \bar{\varepsilon}, \bar{\delta}) \cup \mathrm{int}W(x^j, \bar{\varepsilon}, \bar{\delta})]$. Then S_i' is closed, contains B^i, and $\partial S_i' \cap \partial B^i$ is the set of all x^j with $j \in \mathcal{M}'(x^i)$. Set $S_i = \bigcup_{\{j \in \mathcal{M}:(x^i, x^j) \in C_T(\succsim^*)\}} \mathrm{int}S_j'$. Then S_i is open. Thus, S_i satisfies the requirements. ■

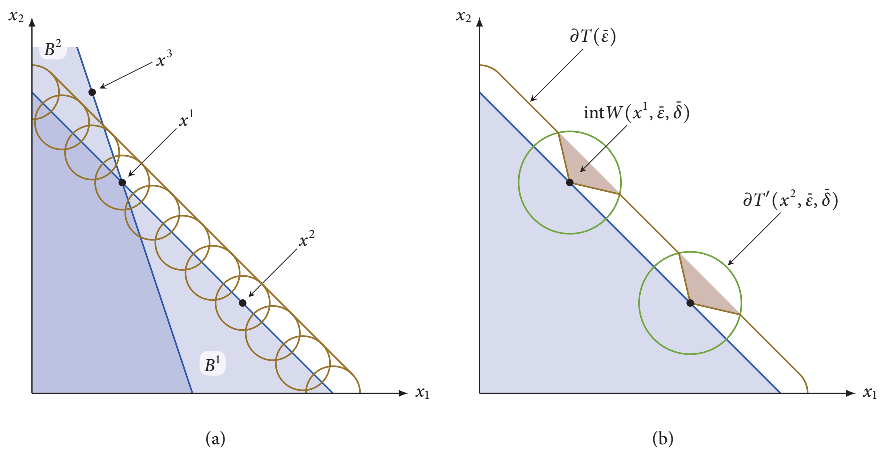


Figure 2: (a): Construction the set $T(\bar{\varepsilon}) = B^1 \cup \bigcup_{x \in \partial B^1} \mathrm{cl}N_{\bar{\varepsilon}}(x)$. (b): Constructing the set $S_1' = T(\bar{\varepsilon}) \setminus \bigcup_{j=1}^{2} [\mathrm{int}T'(x^j, \bar{\varepsilon}, \bar{\delta}) \cup \mathrm{int}W(x^j, \bar{\varepsilon}, \bar{\delta})]$.

An irreflexive and transitive binary relation Q on (a topological space) $\mathbb{X}$ is *lower-semicontinuous* if $\mathcal{L}(Q, x)$ is open for all $x \in \mathbb{X}$.[4] It is *separable* if there exists a countable subset S of $\mathbb{X}$ such that, whenever $(x, y) \in Q$, there exists an $s \in S$ such that $\{(x, s), (s, y)\} \subset Q$. It is *spacious* if whenever $(x, y) \in Q$, then $\mathrm{cl}\mathcal{L}(Q, y) \subset \mathcal{L}(Q, x)$. These definitions are needed for the following theorem.

Theorem 2 (Peleg 1970) *Suppose* Q *is an irreflexive, transitive, lower-semicontinuous, separable, and spacious binary relation on a topological space* $\mathbb{X}$*. Then there exists a continuous real function* $u : \mathbb{X} \to \mathbb{R}$ *such that* $(x, y) \in Q$ *implies* $u(x) > u(y)$*.*

Obviously, such a continuous utility function can be used to define a complete, transitive, and continuous binary relation. Furthermore, if we have such a binary relation, Debreu (1954) has shown that there exists an order preserving utility function, as summarised in the following theorem.

Theorem 3 (Debreu 1954) *Suppose* $Q \in \succsim_T$*. Then there exists a continuous utility function* $u : \mathbb{X} \to \mathbb{R}$ *such that* $(x, y) \in Q$ *implies* $u(x) > u(y)$*.*

Putting these results together, we obtain the following fact.

Fact 5 *For* $Q \in \mathcal{Q}$*, there exists a continuous function* $u : \mathbb{X} \to \mathbb{R}$ *such that* $(x, y) \in Q$ *implies* $u(x) > u(y)$ *if and only if there exists an* $R \in \succsim_T$ *such that* $Q \subseteq A(R)$*.*

Proof The existence of u follows from the fact that R is continuous and Theorem 3. If there exists a continuous u, then $R = \{(x, y) \in \mathbb{X} \times \mathbb{X} : u(x) \geq u(y)\} \in \succsim_T$ satisfies the requirements. ■

We define the following set of rationalising extensions:

$$\mathcal{R}_T = \left\{ Q \in \succsim_T : Q \overset{e}{\to} C_T(\succsim^*) \wedge A(Q) \supset \succ^* \right\}. \tag{28}$$

We can now show that Garp is equivalent to the existence of a complete extension which rationalises Ω, and by Fact 5 the equivalence to the existence of a rationalising utility function follows.

Theorem 4 $\mathcal{R}_T \neq \varnothing$ *if and only if* Ω *satisfies* Garp*.*

Proof To show necessity of Garp, suppose Garp is violated with $(x^j, x^i) \in \succ^*$ and $(x^i, x^j) \in C_C(\succsim^*)$, and assume that Q extends $C_T(\succsim^*)$. Then $(x^j, x^i) \in \succ^*$ implies $(x^j, x^i) \in C_T(\succsim^*)$, and therefore $\{(x^i, x^j), (x^j, x^i)\} \subset Q$. But then $(x^j, x^i) \notin A(Q)$, a contradiction.

To show sufficiency of Garp, suppose Ω satisfies Garp. Let $P(x^i)$ be the set constructed in Lemma 1, i.e. $P(x) = \varnothing$ if $x \notin \Omega_1$ and $P(x^i) = S_i$ otherwise. Note that $x \notin P(x)$. Then $\mathcal{L}(A[C_T(\succsim^*)], x^i) \subset P(x^i)$ and for all $y \in P(x^i)$, $(y, x^i) \notin C_T(\succsim^*)$. Define a binary relation $\succ^+$ on $\mathbb{X}$ as

$$\succ^+ = \{(x, y) \in \mathbb{X} \times \mathbb{X} : y \in P(x)\}.$$

By construction, $\succ^+$ is a binary relation on $\mathbb{X}$ which satisfies the condition in Theorem 2. Thus, by Theorem 2, there exists a continuous utility function u such that $(x, y) \in \succ^+$ implies $u(x) > u(y)$.

[4] In Peleg (1970), Q is said to continuous if $\mathcal{L}(Q, x)$ is open.

If $(x,y) \in S[C_T(\succsim^*)]$, then $P(x) = P(y)$ and $(x,y) \notin \succ^+$. Then we must also have $u(x) = u(y)$: Suppose $u(x) > u(y)$, then by continuity of u, $\{z \in \mathbb{X} : u(z) > u(y)\}$ contains $N_\varepsilon(x)$ for some $\varepsilon > 0$, but $N_\varepsilon(x) \cap P(x) \neq \varnothing$, and with $P(x) = P(y)$, it follows that $\{z \in \mathbb{X} : u(z) > u(y)\} \cap P(y) \neq \varnothing$. But then there is a $z \in P(y)$, therefore $(y,z) \in \succ^+$, and $u(z) > u(y)$, a contradiction.

If $(x,y) \in A[C_T(\succsim^*)]$, then $y \in P(x)$, and therefore $(x,y) \in \succ^+$, which implies $u(x) > u(y)$.

Define $Q = \{(x,y) \in \mathbb{X} : u(x) \geq u(y)\}$. Then $(x,y) \in Q \Leftrightarrow u(x) \geq u(y)$, $(x,y) \in A(Q) \Leftrightarrow u(x) > u(y)$, and $(x,y) \in S(Q) \Leftrightarrow u(x) = u(y)$. Clearly, Q is transitive. With $\succ^+ \subset A(Q)$, $Q \overset{e}{\to} C_T(\succsim^*)$, and (by Fact 3) $\succ^* \subset \succ^+$, the result follows. ■

3.2 *The Problem with Single-Peakedness and Non-Satiation*

With general budget sets, it is not easy to test if there exist non-satiated or single-peaked continuous extensions of a revealed preference relation. This is illustrated in Figure 3, which shows eight budgets and choices. Suppose there is a preference $\succsim$ which is maximised by the choices, and suppose there is a single point of satiation enclosed by the four budgets to the lower left. Consider the possible sets $\mathcal{L}(\precsim, x^i)$ for the four choices to the upper right. These sets either intersect with each other, or $\mathcal{L}(\precsim, x^i)$ intersects with B^i for at least one (x^i, B^i). Hence there have to be at least two points of satiation, both of which are enclosed by four budgets.

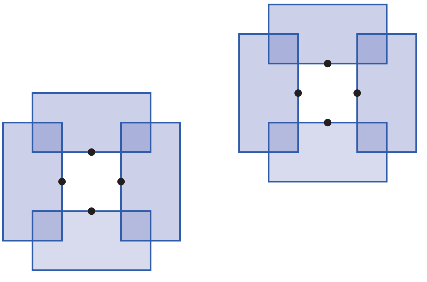


Figure 3: Eight budgets (rectangles) and choices (dots). Even though all budget sets are convex (and therefore contractible) and the set of observations satisfies Garp, there does not exist a transitive, complete, and continuous preference which is also non-satiated or single-peaked.

The budget sets depicted in Figure 3 are all convex, and therefore also contractible (they are homotopically equivalent to a set consisting of a single point). Thus, assuming that budget sets are contractible, "have no holes", or even that budgets are convex is not sufficient to guarantee non-satiation or single-peakedness.

Another interesting result is that while both non-satiation and single-peakedness can be falsified—there exist budgets and choices such that no rationalising non-satiated or single-peaked extension exists, as in Figure 3—, non-satiation cannot be falsified. In particular, whenever there exists a non-satiated rationalising extension, there also exists a single-peaked rationalising extension. Define the following set of rationalising extensions:

$$\mathcal{R}_{NS} = \left\{ Q \in \succsim_{NS} : Q \overset{e}{\to} C_T(\succsim^*) \wedge A(Q) \supset \succ^* \right\}, \tag{29}$$

$$\mathcal{R}_{S(\phi)} = \left\{ Q \in \succsim_{S(\phi)} : Q \overset{e}{\to} C_T(\succsim^*) \wedge A(Q) \supset \succ^* \right\}. \tag{30}$$

Proposition 2 *$\mathcal{R}_{NS} \neq \varnothing$ implies $\mathcal{R}_{S(\phi)} \neq \varnothing$ for some $\phi \in \mathbb{X}$.*

Proof By Theorem 4, Ω satisfies GARP. Suppose first that there exists an $x^i \in \Omega_1$ such that $(x, x^i) \notin C_T(\succsim^*)$ for all $x \neq x^i$. Then, as in the proof of Theorem 4, let $P(x)$ be the set constructed in Lemma 1. Let $P'(x) = P(x)$ if $x \neq x^i$ and $P'(x^i) = \mathbb{X}\backslash\{x^i\}$.

$$\succ^+ = \{(x, y) \in \mathbb{X} \times \mathbb{X} : y \in P'(x)\}.$$

Then the proposition can be proven in analogy to Theorem 4.

Suppose that there does not exist an $x^i \in \Omega_1$ such that $(x, x^i) \notin C_T(\succsim^*)$ for all $x \neq x^i$. Then by GARP there exists a subset $\{x^i\}_{i\in\mathcal{M}'} \subseteq \Omega_1$ with $\mathcal{M}' \subseteq \mathcal{M}$ such that $(x, x^i) \notin C_T(\succsim^*)$ for all $x \notin \{x^i\}_{i\in\mathcal{M}'}$. Then $P(x^i) = P(x^j)$ for all $i, j \in \mathcal{M}'$. Let $x^0 \in N_\varepsilon(x^i)$ for some $i \in \mathcal{M}'$ and some $\varepsilon > 0$ such that $x^0 \notin P(x^k)$ for all $k \in \mathcal{M}$ and $x^0 \notin \Omega_1$ (such a x^0 exists by Lemma 1). Let $P'(x) = P(x)$ if $x \neq x^0$ and $P'(x^0) = \mathbb{X}\backslash\{x^0\}$, and we can apply the same proof as above. ∎

As mentioned in Section 2.4, the definition of $\succ^*$ in Eq. (12) is problematic if the interior of a budget contains a point of satiation: If $x^i \in \text{int}B^i$, then $(x^i, x^i) \in [C_T(\succsim^*) \cap \succ^*]$ and therefore GARP is violated. To account for this possibility, we introduce the following axiom.

Axiom 2 *A set of observations Ω with associated revealed preference relations $(\succsim^*, \succ^*)$ satisfies the* Single-Peaked-Generalised Axiom of Revealed Preference *(SP-GARP) if $(x^i, x^j) \in C_T(\succsim^*)$ and $\Psi[C_T(\succsim^*), x^i] \cap \text{int}B^j \neq \varnothing$ implies $x^i = x^j = \phi$ for one and only one $\phi \in \mathbb{X}$.*

What we would like to establish now is that SP-GARP is necessary and sufficient for the existence of a rationalising relation which is single-peaked at some $x^i \in \Omega_1$, that is, that the following set is non-empty:

$$\mathcal{R}'_{S(\phi)} = \left\{ \mathrm{Q} \in \succsim_{S(\phi)} : \mathrm{Q} \xrightarrow{e} C_T(\succsim^*) \wedge \mathrm{A(Q)} \supset (\succ^* \backslash\{(\phi, \phi)\}) \right\}. \tag{31}$$

Proposition 3 *$\mathcal{R}'_{S(\phi)} \neq \varnothing$ if and only if Ω satisfies SP-GARP.*

Proof Whenever there does not exist a pair $(x^i, x^j) \in C_T(\succsim^*)$ and $(x^j, x^i) \in \succ^*$ (i.e., whenever GARP is satisfied), it follows from Theorem 4 and Proposition 2 that the statement is true.

To show necessity of SP-GARP, suppose SP-GARP is violated with $(x^i, x^j) \in C_T(\succsim^*)$ and $(x^j, x^i) \in \succ^*$, $x^i \neq x^j$, and assume that $\mathrm{Q} \in \mathcal{R}'_{S(\phi)}$. Then we must have $\phi = x^j$, but $(x^i, x^j) \in C_T(\succsim^*)$, so $\mathrm{Q} \notin \succsim_{S(\phi)}$.

To show sufficiency of SP-GARP, suppose SP-GARP is satisfied and $(x^i, x^i) \in \succ^*$. Then $x^i \in \text{int}B^i$. Then we can apply almost the same proof as for Theorem 4, using Lemma 1 with $S_i = \mathbb{X}\backslash\{x^i\}$. ∎

3.3 *Further Extension Results: Convexity and Monotonicity*

We will now impose the assumption that preferences are convex and/or monotonic by applying the appropriate closure operator to the revealed preference relation. To prepare for the rationalisation theorems below, we first need the following algorithm.

For a finite set $S = \{s^i\}_{i=1}^I$ and a binary relation $\mathrm{Q} \subseteq S \times S$, an element $s^i \in S$ is a *maximal element* with respect to Q if $(s^j, s^i) \in \mathrm{Q}$ implies $(s^i, s^j) \in \mathrm{Q}$.

Algorithm 1 (Varian 1982)
Input *A set* $S = \{s^i\}_{i=1}^I$ *indexed by* $\mathcal{I} = \{1, \ldots, I\}$ *and a reflexive binary relation* $\mathrm{Q} \subseteq S \times S$.
Output *An index* $m \in \mathcal{I}$ *which is the index of a maximal element of* S *with respect to* Q.

1. *Set* $m = 1$ *and* $\rho^0 = s^1$.
2. *For each* $i \in \mathcal{I}$,
   - *if* $(s^i, \rho^{i-1}) \in \mathrm{Q}$, *set* $\rho^i = x^i$ *and* $m = i$;
   - *else, set* $\rho^i = \rho^{i-1}$.

Let $MaxE(S, \mathrm{Q})$ be the element returned by Algorithm 1.

Algorithm 2
Input *A set* $S = \{s^i\}_{i=1}^I \subset \mathbb{X}$ *indexed by* $\mathcal{I} = \{1, \ldots, I\}$, *a reflexive binary relation* $\mathrm{Q} \in \mathcal{Q}$, *and a closure operator* $\tilde{C} : \mathcal{Q} \to \mathcal{Q}$.
Output *A partition* $\mathcal{I}^* = \{\mathcal{I}_1^*, \ldots, \mathcal{I}_{I^*}^*\}$ *of* $\mathcal{I}$ *such that all* $i \in \mathcal{I}_\ell^*$ *are maximal elements of* $\bigcup_{k=\ell}^{I^*} \{s^j\}_{j \in \mathcal{I}_k^*}$.

1. *Set* $\mathrm{R} = \mathrm{Q}$. *Set* $m = MaxE(S, \mathrm{R})$. *Set* $\mathcal{I}_1^* = \{i \in \mathcal{I} : (s^i, s^m) \in \mathrm{R}\}$.
2. *If* $\mathcal{I} = \mathcal{I}_1^*$, *set* $I^* = 1$ *and quit. Otherwise, set* $c = 1$ *and go to 3.*
3. *Set* $\mathcal{I}' = \mathcal{I} \setminus \bigcup_{k=1}^c \mathcal{I}_k^*$. *Set* $\mathrm{R} = \tilde{C}(\mathrm{R} \cup \{\{(s^i, s^j)\}_{i \in \mathcal{I}_c^*}\}_{j \in \mathcal{I}'})$.
   Set $m = MaxE(\{x^i\}_{i \in \mathcal{I}'}, \mathrm{R})$. *Set* $\mathcal{I}_{c+1}^* = \{i \in \mathcal{I}' : (s^i, s^m) \in \mathrm{R}\}$.
4. *Set* $c = c + 1$. *If* $\mathcal{I} = \bigcup_{k=1}^c \mathcal{I}_k^*$, *set* $I^* = c$ *and quit. Otherwise, go to 3.*

Algorithm 2, if called with $S = \Omega_1$, $\mathcal{I} = \mathcal{M}$, $\mathrm{Q} = C_C(\succsim^*)$, and $\tilde{C} = C_C$, will give us a partition of the indices of observed choices $\mathcal{M}^*$ such that every subset of $\mathcal{M}^*$ can be interpreted as an equivalence class. According to the DM's revealed preference relation, he is indifferent between every x^i with $i \in \mathcal{M}_k^*$, and his choices do not reveal that he prefers any x^j with $j \in \mathcal{M}_\ell^*$ over any x^i with $i \in \mathcal{M}_k^*$ whenever $k < \ell$.

We would now like to find a condition on the revealed preference relation which is necessary and sufficient for the existence of a rationalising extension of the revealed preference relation which is complete and continuous and which is also (i) convex, (ii) convex and single-peaked at a certain point $\phi \in \mathbb{X}$, (iii) monotonic, and (iv) both convex and monotonic.

3.3.1 *Convexity*

We start with convexity. Let

$$\mathcal{R}_C = \left\{ \mathrm{Q} \in \succsim_C : \mathrm{Q} \overset{e}{\to} C_C(\succsim^*) \wedge \mathrm{A}(\mathrm{Q}) \supset \succ^* \right\}, \tag{32}$$

be the set of all rationalising binary relations which are convex.

Axiom 3 *A set of observations* Ω *with associated revealed preference relations* $(\succsim^*, \succ^*)$ *satisfies the* Convexity-Generalised Axiom of Revealed Preference *(C-GARP) if* $CH(\Psi[C_C(\succsim^*), x^i]) \cap \mathrm{int}B^i = \varnothing$ *for all* $i \in \mathcal{M}$.

Theorem 5 $\mathcal{R}_C \neq \varnothing$ *if and only if* Ω *satisfies C-GARP.*

Proof To show necessity of C-GARP, suppose C-GARP is violated with $y \in CH(\Psi[C_C(\succsim^*), x^i]) \cap \mathrm{int}B^i$, and assume that Q extends $C_C(\succsim^*)$. By Fact 1 $C_C(\succsim^*)$ is convex, thus $(y, x^i) \in C_C(\succsim^*)$. But with $y \in \mathrm{int}B^i$, $(x^i, y) \in \succ^*$, therefore $\{(y, x^i), (x^i, y)\} \subset \mathrm{Q}$, and therefore $(x^i, y) \notin \mathrm{A}(\mathrm{Q})$.

To show sufficiency of C-Garp, suppose C-Garp is satisfied. We then proceed with the following steps.

Step 1 Let $\mathcal{M}^*$ be the partition of $\mathcal{M}$ found by calling Algorithm 2 with the set $S = \Omega_1$ indexed by $\mathcal{I} = \mathcal{M}$, $\mathrm{Q} = C_C(\succsim^*)$, and $\tilde{C} = C_C$. Let M^* be the cardinality of $\mathcal{M}^*$. Let $\mathcal{M}^*_{<i} = \bigcup_{j=1}^{i-1} \mathcal{M}^*_j$, $\mathcal{M}^*_{\leq i} = \bigcup_{j=1}^{i} \mathcal{M}^*_j$, $\mathcal{M}^*_{>i} = \bigcup_{j=i+1}^{M*}$, and $\mathcal{M}^*_{\geq i} = \bigcup_{j=i}^{M*}$.

Whenever $(x^i, x^j) \in \mathrm{A}[C_C(\succsim^*)]$ then $i \in \mathcal{M}^*_m$ for some m and $j \in \mathcal{M}^*_{>m}$, and whenever $(x^i, x^j) \in \mathrm{S}[C_C(\succsim^*)]$ then $\{i, j\} \subseteq \mathcal{M}^*_m$. If $\mathcal{M}^*_1$ contains only a single element i, let $\phi = x^i$. We omit the full proof for the case in which $\mathcal{M}^*_1$ contains more than one element; but in that case, a ϕ can be found either in the non-empty interior of the convex hull of all $\{x^i\}_{i \in \mathcal{M}^*_1}$ or in an ε-neighbourhood of some x^i.

By definition, for all $i \in \mathcal{M}^* m$, either (a) $B^i \cap CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}}) \neq \{x^i\}$, or (b) $x^i \in \partial CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}})$ and x^i is a convex combination of some $x^j, x^k, \ldots$ with $\{i, j, \ldots\} \subset \mathcal{M}^*_{\leq m}$, or (c) $i \in \mathcal{M}^*_\ell$, $\ell \leq m$, and $\{x^j\}_{j \in \mathcal{M}^*_\ell}$ consists of only element which is not a convex combination of elements in $\{x^k\}_{k \in \mathcal{M}^*_{\leq \ell}}$. In case (b), x^i is not an extreme point (i.e., $\{x^i\}$ is not a vertex) of $CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}})$; thus $CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}}) = CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}} \setminus \{x^i\})$.

Step 2 We will now show that at every step of Algorithm 2, the updated relation R satisfies C-Garp in the sense that $CH[\Psi(R, x^i)] \cap \mathrm{int} B^i \neq \varnothing$. We will show that this is the case if and only if Ω satisfies C-Garp. That is, $CH[\Psi(R, x^i)] \cap \mathrm{int} B^i \neq \varnothing$ if and only if $CH(\Psi[C_C(\succsim^*), x^i]) \cap \mathrm{int} B^i \neq \varnothing$. It is obvious that R cannot satisfy C-Garp if Ω violates C-Garp. We will show sufficiency; so suppose Ω satisfies C-Garp.

Let $\mathrm{R}_1 = C_C(\succsim^*)$ and let R_m, $m \geq 2$, be the relation R after the third step of the algorithm has been executed for the $(m-1)$th time. Then $CH[\Psi(\mathrm{R}_m, x^i)] = CH(\{x^k\}_{k \in \mathcal{M}^*_{\leq m}})$. It is obvious that R_1 satisfies C-Garp. For R_{m-1}, set $y = x^j$ and $S = \{x^i\}$ with $j \in \mathcal{M}^*_{m-1}$ and $i \in \mathcal{M}^*_m$. Then $\Upsilon(S, \mathrm{R}_{m-1}, y) = CH[\{x^j\} \cup \mathcal{U}(\mathrm{R}_{m-1}, x)] = CH(\{x^k\}_{k \in \mathcal{M}^*_{<m}} \cup \{x^i\})$. If $\Upsilon(S, \mathrm{R}_{m-1}, y) \cap \mathrm{int} B^i \neq \varnothing$, then $\Upsilon^*(S, \mathrm{R}_{m-1}, y) \cap \mathcal{L}(\mathrm{R}_{m-1}, x^i) \neq \{x^i\}$ and hence $(x^i, y) \in \mathrm{R}_{m-1}$. As $y = x^j$ with $j \in \mathcal{M}^*_{m-1}$, this implies that $i \in \mathcal{M}^*_{m-1}$, a contradiction.

Step 3 We will now construct a sequence of compact convex sets $\Theta_1 \subset \Theta_2 \subset \ldots \Theta_{M^*}$ which are used to define preferred and worse sets. As $\phi \in \Omega_1$, let $\Theta_1 = \{\phi\}$. Let $HC(x, \varepsilon)$ be the L-dimensional hypercube with centre x and volume ε^L. Set $\Theta_2 = CH[\{x^i\}_{i \in \mathcal{M}^*_2} \cup HC(\phi, \varepsilon)]$.

Let V_i denote the set of vertices of Θ_i. Define recursively for $j \in \{3, \ldots, M^*\}$

$$\Theta_j = CH\Big[\{x^i\}_{i \in \mathcal{M}^*_j} \cup \bigcup_{v \in V_{j-1}} HC(v, \varepsilon)\Big].$$

Then define for $j \in \{1, \ldots, M^* - 1\}$ and $\lambda \in [0, 1]$

$$\Pi_j(\lambda) = \lambda\, \Theta_j + (1 - \lambda)\Theta_{j+1}.$$

All Θ_j and $\Pi_j(\lambda)$ are convex polytopes with finitely many vertices, and $\Theta_1 \subset \Theta_2 \subset \ldots \Theta_{M^*}$. See Figure 4 for an example.

The sets Θ_m are our candidates for $\mathcal{U}(\mathrm{Q}, x^i)$ for some $\mathrm{Q} \in \mathcal{R}_C$ and all $i \in \mathcal{M}^*_m$. The sets Θ_m are compact and convex by construction. We need to show that there exist $\varepsilon > 0$ small enough such that (i) $x^i \in \partial\Theta_m$ for all $i \in \mathcal{M}^*_m$, (ii) $x^i \in \mathrm{int}\Theta_m$ for all $i \in \mathcal{M}^*_{<m}$, (iii) $x^i \notin \Theta_m$ for all $i \in \mathcal{M}^*_{>m}$, and (iv) $\mathrm{int} B^i \cap \Theta_m = \varnothing$ for all $i \in \mathcal{M}^*_m$.

(i) In a slight abuse of previous definitions, we have $\Theta_m = CH(\{x^j\}_{j\in\mathcal{M}^*_{\leq m}})$ for $\varepsilon = 0$. By construction, if $x^i \notin \partial\Theta_m$ for all $\varepsilon > 0$ then either $x^i \in \text{int}CH(\{x^j\}_{j\in\mathcal{M}^*_{\leq m}})$ or $x^i \in \partial CH(\{x^j\}_{j\in\mathcal{M}^*_{\leq m}})$ but x^i is a convex combination of some $x^j, x^k, \ldots$ with $\{i, j, \ldots\} \subset \mathcal{M}^*_{\leq m}$.
In both cases, there is at least one x^j with $j \in \mathcal{M}^*_m$ with $i \neq j$ such that x^j is not a convex combinations of elements with indices in $\mathcal{M}^*_{\leq m}$. Thus, $(x^j, x^k) \in \text{R}$ with $k \in \mathcal{M}^*_{<m}$, where R is the relation computed in Algorithm 2. But that contradicts $j \in \mathcal{M}^*_m$.

(ii) By construction $x^i \notin \text{int}\Theta_m$ implies $x^i \in \partial\Theta_m$. But by (i), $x^i \in \partial\Theta_\ell$, $\ell < m$. But then for any $\varepsilon > 0$, $x^i \in \text{int}\Theta_m$.

(iii) Similar to (i): $x^i \in \Theta_m$ contradicts $i \in \mathcal{M}^*_{>m}$.

(iv) Given (i), if $\text{int}B^i \cap \Theta_m \neq \varnothing$ for all $\varepsilon > 0$, then $\text{int}B^i \cap CH(\{x^j\}_{j\in\mathcal{M}^*_{\leq m}})$. But then R violates C-Garp, which contradicts the finding in Step 2.

Step 4 For $j = 1, \ldots, M^*$, let

$$\begin{aligned}
\succsim^{j,\lambda} &= \{(x,y) \in \mathbb{X}\times\mathbb{X} : x \in \Pi_j(\lambda) \wedge y \notin \Pi_j(\lambda)\backslash\partial\Pi_j(\lambda)\},\\
\succsim^{M^*+1,\lambda} &= \{(x,y) \in \mathbb{X}\times\mathbb{X} : [x \in \Theta_{M^*} \wedge y \notin \Theta_{M^*}] \vee [x, y \notin \Theta_{M^*} \wedge \min_{z\in\Theta_{M^*}} d(x,z) \leq \min_{z\in\Theta_{M^*}} d(y,z)]\}.
\end{aligned}$$

Then let

$$\begin{aligned}
\succsim^+ &= \left(\bigcup_{j=1}^{M^*+1} \bigcup_{\lambda\in[0,1]} \succsim^{j,\lambda}\right) \cup \left(\bigcup_{x\in\mathbb{X}} \{(\phi, x)\}\right)\\
\succ^+ &= \text{A}(\succsim^+).
\end{aligned}$$

Because all $\Pi_j(\lambda)$ are closed convex sets, $\succsim^+ \in \succsim_C$ and, by construction, $C_C(\succsim^*) \subset \succsim^+$, and $(x,\phi) \in \succsim^+$ implies $x = \phi$. It remains to be shown that $\text{A}(\succsim^+) \supset \succ^*$ and we have $\succsim^+ \in \mathcal{R}_C$. If $(x,y) \in \succ^*$, then $x = x^i \in \Omega_1$ and $y \in \text{int}B^i$. Then if $(x^i, y) \notin \text{A}(\succsim^+)$, we have $(y, x^i) \in \succsim^+$. But then for $i \in \mathcal{M}^*_m$, we have $\text{int}B^i \cap \Theta_m \neq \varnothing$ which contradicts the findings in Step 3. ■

We can use the result to account for the possibility that the point of satiation of a single-peaked preference is in the interior of a budget set. Let

$$\mathcal{R}_{C,S(\phi)} = \left\{\text{Q} \in (\succsim_C \cap \succsim_{S(\phi)}) : \text{Q} \overset{e}{\to} C_C(\succsim^*) \wedge \text{A}(\text{Q}) \supset (\succ^* \backslash\{(\phi,\phi)\})\right\}. \tag{33}$$

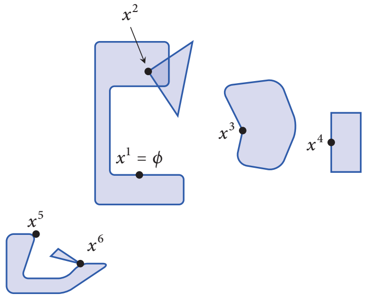
Axiom 4 *A set of observations Ω with associated revealed preference relations $(\succsim^*, \succ^*)$ satisfies the* Convexity-Single-Peaked-Generalised Axiom of Revealed Preference *(C-Sp-Garp) if $(x^i, x^j) \in C_C(\succsim^*)$ and $CH(\Psi[C_T(\succsim^*), x^i]) \cap \text{int}B^j \neq \varnothing$ implies $x^i = x^j = \phi$ for one and only one $\phi \in \mathbb{X}$.*

Then from Theorem 5, we obtain the following interesting corollary.

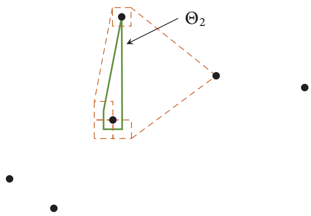
Corollary 1 *$\mathcal{R}_{C,S(\phi)} \neq \varnothing$ if and only if Ω satisfies C-Sp-Garp.*

Proof Necessity of C-Sp-Garp can be shown in analogy to necessity of C-Garp in the proof of Theorem 5.

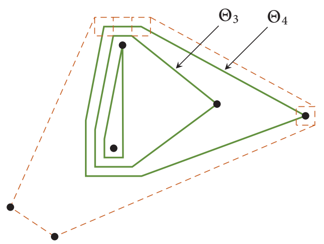
Note that $(x^i, x^i) \in C_C(\succsim^*)$. Then if C-Sp-Garp is satisfied, then a violation of C-Garp can only occur of $x^i \in \text{int}B^i$, and for all x^j with $(x^j, x^i) \in C_C(\succsim^*)$, $x^j = x^i$. Then the point of satiation is $\phi = x^i$. In the



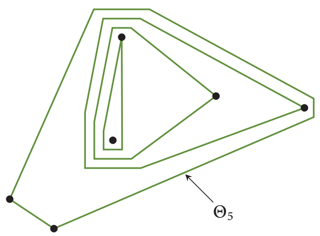
(a) A two dimensional example: The observations.



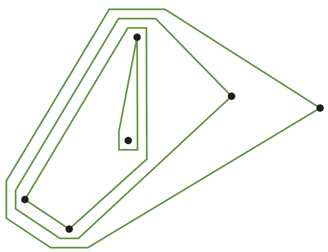
(b) Θ_2 and the construction of Θ_3.



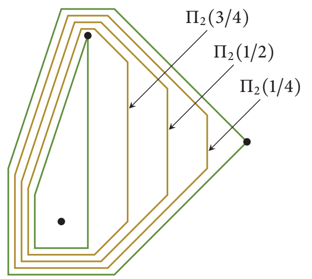
(c) Θ_2, Θ_3, Θ_4, and the construction of Θ_5.



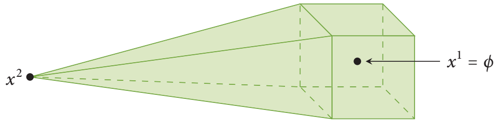
(d) All Θ_i.



(e) An alternative assignment.



(f) A closer look: $\Pi_2(\lambda)$ as weighted Minkowski averages



(g) A three dimensional example.

Figure 4

proof of Theorem 5, we have actually constructed a $Q \in \mathcal{R}_{C,S(\phi)}$; it can be easily extended to show that if C-Sp-Garp is satisfied but C-Garp is violated, the construction still works. ■

3.3.2 *Monotonicity*

Let

$$\mathcal{R}_M = \left\{ Q \in \gtrsim_M : Q \xrightarrow{e} C_M(\gtrsim^*) \wedge A(Q) \supset \succ^* \right\}. \tag{34}$$

Axiom 5 *A set of observations Ω with associated revealed preference relations $(\gtrsim^*, \succ^*)$ satisfies the* Monotonicity-Generalised Axiom of Revealed Preference *(M-Garp) if $MH(\Psi[C_M(\gtrsim^*), x^i]) \cap \text{int}B^i = \varnothing$ for all $i \in \mathcal{M}$.*

Theorem 6 *$\mathcal{R}_M \neq \varnothing$ if and only if Ω satisfies M-Garp.*

We omit the proof here, as it is similar to the proof of Theorem 5. Sufficiency of M-Garp is again quite obvious. Sufficiency of M-Garp can be shown in analogy to the proof of Theorem 5 based on a slightly modified version of Alogrithm 2 and replacing convex hulls with monotonic hulls.

3.3.3 *Convexity and Monotonicity*

We can combine convexity and monotonicity to obtain a rationalisation result which corresponds to Afriat's Theorem. Let

$$\mathcal{R}_{C,M} = \left\{ Q \in (\gtrsim_C \cap \gtrsim_M) : Q \xrightarrow{e} C_M(\gtrsim^*) \wedge A(Q) \supset \succ^* \right\}. \tag{35}$$

Axiom 6 *A set of observations Ω with associated revealed preference relations $(\gtrsim^*, \succ^*)$ satisfies the* Convexity-Monotonicity-Generalised Axiom of Revealed Preference *(C-M-Garp) if $CMH(\Psi[C_{C,M}(\gtrsim^*), x^i]) \cap \text{int}B^i = \varnothing$ for all $i \in \mathcal{M}$.*

Theorem 7 *$\mathcal{R}_{C,M} \neq \varnothing$ if and only if Ω satisfies C-M-Garp.*

We omit the full proof here. Necessary of C-M-Garp can be shown in analogy to the previous proofs. Sufficiency of C-M-Garp can be shown with a variant of the proofs of Theorems 5 and 6, by replacing convex and monotonic hulls with convex monotonic hulls.

3.4 *Restrictions on Budgets: Competitive Consumers*

In consumer theory with competitive budgets from the $\mathcal{B}_D$ family, Afriat's Theorem shows that Garp is not only equivalent to the existence of a rationalising non-satiated utility function, but also the existence of a rationalising monotonic and concave utility function.

Theorem 8 (Afriat 1967, Diewert 1973, Varian 1982) *Suppose* $\Omega = \{(x^i, B^i)\}_{i\in\mathcal{M}} \subset \mathbb{X} \times \mathcal{B}_D$. *The following conditions are equivalent:*

(i) The set of observations Ω *satisfies* GARP.

(ii) There exists a non-satiated continuous utility function which rationalises Ω.

(iii) There exists a monotonic, concave, and continuous utility function which rationalises Ω.

With the results we have obtained so far we obtain the following corollary, which shows that GARP and C-M-GARP are equivalent when all budgets are competitive. It also shows that GARP implies the existence of a rationalising single-peaked extension.

Corollary 2 *Suppose* $\Omega = \{(x^i, B^i)\}_{i\in\mathcal{M}} \subset \mathbb{X} \times \mathcal{B}_D$. *The following conditions are equivalent:*

(i) The set of observations Ω *satisfies* GARP.

(ii) The set of observations Ω *satisfies C-M-*GARP.

(iii) There exists a $Q \in \succsim_{NS}$, *with* $Q \overset{e}{\to} C_T(\succsim^*)$ *and* $A(Q) \supset \succ^*$.

(iv) There exists a $Q \in (\succsim_M \cap \succsim_C)$, *with* $Q \overset{e}{\to} C_{C,M}T(\succsim^*)$ *and* $A(Q) \supset \succ^*$.

Furthermore, GARP *implies that there exists a* $Q \in \succsim_{S(\phi)}$, *with* $Q \overset{e}{\to} C_T(\succsim^*)$ *and* $A(Q) \supset \succ^*$.

Proof If there exists a rationalising utility function u, we can let $Q = \{(x, y) \in \mathbb{X} \times \mathbb{X} : u(x) \geq u(y)\}$, and $Q \overset{e}{\to} C_{C,M}(\succsim^*)$ and $A(Q) \supset \succ^*$ follows. Then (i) ⇒ (ii) follows from Theorem 8; (ii) ⇒ (i) is obvious. (ii) ⇔ (iv) follows from Theorem 7. (i) ⇔ (iii) follows from Theorem 8. The final statement follows from Proposition 2. ■

4 RECOVERABILITY AND COMPUTATIONAL ASPECTS

4.1 *Recoverability: Nonparametric Analysis of Choice Data*

As Varian (1982) has shown, revealed preference relations can be used to recover a large part of a consumer's preference when he is observed to choose from competitive budget sets and satisfies GARP. This nonparametric approach allows to recover large parts of a preference without relying on specific functional forms of utility. Varian defines the "revealed preferred set" in terms of GARP. The set of price vectors which support a consumption bundle x is the set of all prices at which x can be demanded without violating GARP, given all the observed choices. The the revealed preferred set of some $y \in \mathbb{X}$ is the set of all bundles which, when demanded at any supporting price vector, will be (strictly) revealed preferred to y. T

Varian (1982) showed that the interior of the convex monotonic hull of all choices revealed preferred to x is a subset of the revealed preferred set of x. Knoblauch (1992) has shown that the revealed preferred set of x is a subset of the closure of the convex monotonic hull of all choices revealed preferred to x. The definition of the revealed preferred set in terms of GARP and in the context of competitive budget sets implicitly assumes that preferences are both convex and monotonic. GARP guarantees that this hypothesis cannot be rejected, as Afriat's Theorem shows. We would like to make this assumption more explicit in the following proposition, which is based on the intersection of all rationalising extensions of the revealed preference relation which satisfy additional assumptions.

Suppose Ω satisfies C-GARP; then we know that there exists a $Q \in \mathcal{R}_C$. We would like to know, for some arbitrary element $x \in \mathbb{X}$, the set of all elements in $\mathbb{X}$ such that *every* $Q \in \mathcal{R}_C$ ranks this element higher than x; similarly, we would like to know the set of all elements which a ranked lower than x by every

$Q \in \mathcal{R}_C$. Preferably, these sets should be described in a way that makes their construction operational. Proposition 4 provides such a way. Similarly to Varian (1982) and Knoblauch (1992), we show that these sets are characterised by the convex hull of a subset of the observed choices Ω_1. The proposition furthermore distinguishes between convex and monotonic extensions.

Proposition 4 *Suppose Ω satisfies (1) C-Garp, (2) C-Sp-Garp, (3) M-Garp, (4) C-M-Garp. Then*

$$\text{int}CH(\Psi[C_C(\succsim^*), x] \cup \{x\}) \subseteq \bigcap_{Q \in \mathcal{R}_C} \mathcal{U}(Q, x) \subseteq CH(\Psi[C_C(\succsim^*), x] \cup \{x\}) \quad \text{(Prop. 4.1)}$$

$$\text{int}CH(\Psi[C_{C,S(\phi)}(\succsim^*), x] \cup \{x\}) \subseteq \bigcap_{Q \in \mathcal{R}_{C,S(\phi)}} \mathcal{U}(Q, x) \subseteq CH(\Psi[C_{C,S(\phi)}(\succsim^*), x] \cup \{x\}) \quad \text{(Prop. 4.2)}$$

$$\text{int}MH(\Psi[C_M(\succsim^*), x] \cup \{x\}) \subseteq \bigcap_{Q \in \mathcal{R}_M} \mathcal{U}(Q, x) \subseteq MH(\Psi[C_M(\succsim^*), x] \cup \{x\}) \quad \text{(Prop. 4.3)}$$

$$\text{int}CMH(\Psi[C_{C,M}(\succsim^*), x] \cup \{x\}) \subseteq \bigcap_{Q \in \mathcal{R}_{C,M}} \mathcal{U}(Q, x) \subseteq CMH(\Psi[C_{C,M}(\succsim^*), x] \cup \{x\}) \quad \text{(Prop. 4.4)}$$

Proof We only proof Prop. 4.1 here; the other proofs are quite similar. If $y \in \text{int}CH(\Psi[C_C(\precsim^*), \tilde{x}] \cup \{\tilde{x}\})$, then $(y, x^i) \in Q$ for all $Q \in \succsim_C$ for which $Q \overset{e}{\to} C_C(\succsim^*)$, thus $y \in \mathcal{U}(Q, \tilde{x})$.

Suppose $y \notin CH(\Psi[C_C(\succsim^*), \tilde{x}] \cup \{\tilde{x}\})$. We need to show that there exists a $Q \in \mathcal{R}_C$ such that $y \notin \mathcal{U}(Q, \tilde{x})$. Let $\tilde{\Upsilon} = \Upsilon^*(\Psi[C_C(\succsim^*), \tilde{x}], C_C(\succsim^*), \tilde{x})$. Then, with $x^k \in \Psi[C_C(\succsim^*), \tilde{x}]$, it follows from the definition of the convex closure that either (i) $\tilde{\Upsilon} \cap B^k \neq \{x^k\}$, or (ii) $\{x^k\}$ is not a vertex of $CH(\Psi[C_C(\succsim^*), \tilde{x}])$ which implies $\tilde{\Upsilon} = \Upsilon^*(\Psi[C_C(\succsim^*), \tilde{x}] \setminus \{x^k\}, C_C(\succsim^*), \tilde{x})$, or (iii) $x^\ell \in \Psi[C_C(\succsim^*), \tilde{x}]$, $(x^k, x^\ell) \in C_C(\succsim^*)$, and $\tilde{\Upsilon} \cap B^\ell \neq \{x^\ell\}$. But then by the definition of Υ^*, we have $\tilde{\Upsilon} = \Upsilon^*[S, C_C(\succsim^*), \tilde{x}]$ with $S = \{x^i\}_{\{i \in \mathcal{M}: \tilde{\Upsilon} \cap B^i \neq \{x^i\}\}}$.

Let V be the set of all vertices of $CH(\Psi[C_C(\succsim^*), \tilde{x}])$. Let $B^{M+1} = \mathbb{X} \setminus CH[\bigcup_{v \in V} HC(v, \varepsilon)]$ with $\varepsilon > 0$ small enough such that $y \in B^{M+1}$, $\tilde{x} \in \partial B^{M+1}$, $\mathbb{X} \setminus B^{M+1}$ is convex, and $x^j \in B^{M+1}$ for all $x^j \notin \Psi[C_C(\succsim^*), \tilde{x}]$. Let $x^{M+1} = \tilde{x}$ and $\tilde{\Omega} = \Omega \cup \{(x^{M+1}, B^{M+1})\}$, and let $(\tilde{\succsim}^*, \tilde{\succ}^*)$ be the revealed preference relation associated with $\tilde{\Omega}$.

We need to show that $\tilde{\Omega}$ satisfies C-Garp. Clearly the subset of $\tilde{\Omega}$ restricted to $\Psi[C_C(\succsim^*), \tilde{x}]$ and x^{M+1} satisfies C-Garp. For all $x^\ell \notin \Psi[C_C(\succsim^*), \tilde{x}]$, $\tilde{\Upsilon} \subset CH(\Psi[C_C(\tilde{\succsim}^*), x^\ell])$. Thus, if $CH(\Psi[C_C(\tilde{\succsim}^*), x^\ell]) \cap \text{int}B^\ell \neq \emptyset$, then with $S = \{x^i\}_{\{i \in \mathcal{M}: \tilde{\Upsilon} \cap B^i \neq \{x^i\}\}} \cup \{x^\ell\}$ we have $\Upsilon^*[S, C_C(\succsim^*), \tilde{x}] \cap B^j \neq \{x^j\}$ for all $x^j \in S$. Thus $(x^\ell, \tilde{x}) \in C_C(\succsim^*)$, which contradicts $x^\ell \notin \Psi[C_C(\succsim^*), \tilde{x}]$. Thus, using the same steps as in the proof of Theorem 5, we can construct a $Q \in \mathcal{R}_C$ such that $y \notin \mathcal{U}(Q, \tilde{x})$.

■

See Figure 5 for an example of Prop. 4.1. By definition of the sets $\mathcal{U}$ and $\mathcal{L}$, we have that $y \in \mathcal{L}(Q, x)$ if and only if $x \in \mathcal{U}(Q, y)$; thus, Proposition 4 also covers the set of all elements which a ranked lower than x by every $Q \in \mathcal{R}_C$.

4.2 *Computational Aspects*

The most important results of the paper are based on relations which are restricted to observed choices. The axioms can be tested by only comparing the relations between the elements in Ω_1, which is finite. The sets considered in Proposition 4 only consider convex and monotonic hulls of subsets of Ω_1. The analysis is therefore, in principal, operational.

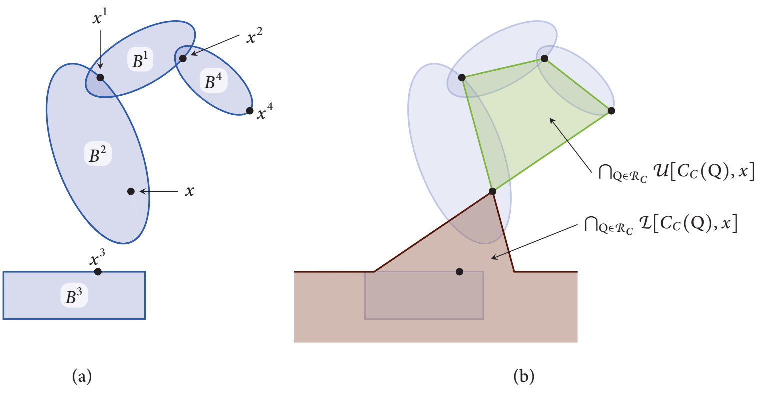


Figure 5: (a): Four observations and some point x. (b): The set of all $y \in \mathbb{X}$ which, for every convex extension of $C_C(\succsim^*)$, must be preferred to x, and the set of all $y \in \mathbb{X}$ which must be worse than x.

When choices are observed on competitive budget sets, it is very easy to compute the revealed preference relation, as budgets are merely hyperplanes implicitly defined by the price vector. The general setup, where budgets are compact subsets of $\mathbb{X}$ with non-empty relative interiors, complicates this computation. When budgets can be described by a continuous function $g : \mathbb{X} \to \mathbb{R}$, such that $B = \{x \in \mathbb{X} : g(x) \geq 0\}$, such as the budgets considered by Forges and Minelli (2009), it is quite straightforward to compute $\succsim^*$ and its closures.

Experiments based on the framework in this paper are naturally limited to budgets which can be presented to subjects in an understandable way. The graphical approach introduced by Choi et al. (2007a) and Fisman et al. (2007), which allows subjects to click on their desired element with a computer mouse, certainly allows more general budget sets than competitive ones. Simple non-monotonic budget sets, such as those considered in Andreoni and Miller (2002) to test preferences for inequality aversion, can be easily described both in mathematical terms and graphically. Thus, budgets do not need to be overly complicated to generate useful extra information which cannot be collected using competitive budget sets. What design of budget is useful depends on the context of the study, but the analysis in this paper covers almost every conceivable possibility.

5 OTHER CLOSURE OPERATORS

5.1 *Mean-Variance and Stochastic Dominance*

We first consider a portfolio choice framework, which is based on Heufer (2011).

Let $\mathbb{X} = \mathbb{R}^L_+$. There are L different states which can obtain after the portfolio choice has been made. In each state $i \in \{1, \ldots, L\}$, asset i is the only asset that pays off. State i occurs with probability $\pi_i \in \Delta$, where Δ is the $(L-1)$ probability simplex, i.e., $\pi_i \geq 0$ for all i and $\sum_{i=1}^{L} \pi_i = 1$. The probability vector π is known to the DM. Let $\tilde{\pi}(x)$ denote the ex post realised payoff of a portfolio x.

Note that the asset space $\mathbb{X}$ and the space of state contingent payoffs are the same. A portfolio $x = (x_1, \ldots, x_L) \in \mathbb{X}$ specifies the amounts invested in L different assets, where an asset is a state-contingent claim. We can define an asset as a column vector $X_{\cdot,i} = (X_{1,i}, \ldots, X_{L,i})'$ which specifies the payoff in the different states $1, \ldots, L$, and x_i is the amount of money invested in this asset. In the present framework, asset i is simply given by $X_{i,i} = 1$ and $X_{j,i} = 0$ for $j \neq i$, and X is the identity matrix. These basic assets are also known as Arrow-Debreu securities. The payoff in state j of a portfolio x is then $(X_{j,\cdot})x = x_j$. Instead of defining the DM's preferences over payoffs in the different states, we can equivalently define the preferences over portfolios.

This setup is more general than it may appear: Suppose that instead of Arrow-Debreu securities, there are $K \geq 2$ linearly independent general assets $Y_{\cdot,i}$. Asset $Y_{\cdot,i}$ pays off $Y_{j,i} \geq 0$ in state j. If we allow short-selling of these assets, that is, to invest in a negative amount of some of the general assets, we need a no arbitrage (no free lunch) condition. If this condition holds, and if there are at least $K = L$ linearly independent general assets, then the problem of choosing a portfolio of general assets is isomorphic to a problem of choosing a portfolio of Arrow-Debreu securities. Then if instead of choices over basic Arrow-Debreu securities we observe choices over more general assets, we can transform the observations into an equivalent set of observations over (fictional) Arrow-Debreu securities. This follows from the work of Ross (1978), Breeden and Litzenberger (1978), and Varian (1987), among others.

Let $\mathrm{E}(x) = \sum \pi_i x_i$ be the expected value of a portfolio $x \in \mathbb{X}$. Let $\succsim_E \in \mathcal{Q}$ be defined as

$$\succsim_{\mathrm{E}} = \{(x, y) \in \mathbb{X} \times \mathbb{X} : \mathrm{E}(x) \geq \mathrm{E}(y)\} \tag{36}$$

We will drop the π if the reference is clear. Let $\sim_E = \mathrm{S}(\succsim_E)$ and $\succ_E = \mathrm{A}(\succsim_E)$.

The variance of a portfolio is $\mathrm{Var}(x) = \mathrm{E}[(x)^2] - \mathrm{E}(x)^2$, where $(x)^2 = (x_1^2, \ldots, x_L^2)$. Let $\succsim_{\mathrm{Var}} \in \mathcal{Q}$ be defined as

$$\succsim_{\mathrm{Var}} = \{(x, y) \in \mathbb{X} \times \mathbb{X} : \mathrm{Var}(x) \leq \mathrm{Var}(x)\} \tag{37}$$

Let $F : \mathbb{R} \times \mathbb{X} \times \Delta \to [0,1]$ be the cumulative distribution function of a portfolio, i.e., $F(\xi, x, \pi) = \mathrm{Prob}(\tilde{\pi}(x) \leq \xi)$ gives the probability that the payoff from a portfolio $x \in \mathbb{X}$ is less than or equal to $\xi \in \mathbb{R}$. Let $\xi^i \in \mathbb{R}_+$, for $i = 1, \ldots, n \leq 2L$, be one of the payoffs of two portfolios x and y, i.e., $\xi^i \in \{x_1, \ldots, x_L\} \cup \{y_1, \ldots, y_L\}$, sorted in increasing order, with n denoting the number of distinct x_i and y_i. That is, when we compare any two portfolios x and y, an ξ^i is one of the ex post payoffs; with two portfolios there is at least one distinct ex post payoff and there are at most $2L$ distinct payoffs. Then let $\succsim_{\mathrm{FSD}}$ and $\succsim_{\mathrm{SSD}}$ be binary relations on $\mathbb{X}$, defined as

$$\succsim_{\mathrm{FSD}} = \{(x, y) \in \mathbb{X} \times \mathbb{X} : \forall \xi^i F(\xi^i, x, \pi) \leq F(\xi^i, y, \pi)\},$$

$$\succsim_{\mathrm{SSD}} = \{(x, y) \in \mathbb{X} \times \mathbb{X} : \forall \ell < n, \forall \xi^i, \sum_{i=1}^{\ell} F(\xi^i, x, \pi)[\xi^{i+1} - \xi^i] \leq \sum_{i=1}^{\ell} F(\xi^i, y, \pi)[\xi^{i+1} - \xi^i]\}.$$

The relations are called the *first and second order stochastic dominance* relations, respectively (see Hadar and Russell 1969): x has *first order stochastic dominance* (FSD) over y if $(x, y) \in \succsim_{\mathrm{FSD}}$, and *second order stochastic dominance* (SSD) if $(x, y) \in \succsim_{\mathrm{SSD}}$. Suppose x has FSD (SSD) order stochastic dominance over y. Then every expected utility maximiser with a monotonically increasing (and concave) utility function will prefer x over y (see, for example, Hanoch and Levy 1969).

Note that $(x, y) \in \succsim_{\mathrm{SSD}}$ and $(y, x) \in \succsim_{\mathrm{SSD}}$ if and only if $F(\xi^i, x) = F(\xi^i, y)$. Thus, $(x, y) \in \succ_{\mathrm{SSD}}$ if and only if $(x, y) \in \succsim_{\mathrm{SSD}}$ and $F(\xi^i, x) \neq F(\xi^i, y)$. The same is true for $\succsim_{\mathrm{FSD}}$.

For mean-variance preferences, we assume that a DM prefers portfolio x over portfolio y whenever x has a higher expected value and a lower variance, that is $\succsim \supset (\succsim_E \cap \succsim_{\text{Var}})$. If a DM maximises a monotonically increasing utility function defined on $\mathbb{X}$, it can be easily shown that $\succsim \supset \succsim_{\text{FSD}}$, and for a risk averse expected utility maximiser, $\succsim \supset \succsim_{\text{SSD}}$. Note that $(\succsim_E \cap \succsim_{\text{Var}}) \subset \succsim_{\text{SSD}}$ and $\succsim_{\text{FSD}} \subset \succsim_{\text{SSD}}$, but $(\succsim_{\text{E}} \cap \succsim_{\text{Var}}) \notin \succsim_{\text{FSD}}$ and $\succsim_{\text{FSD}} \notin (\succsim_{\text{E}} \cap \succsim_{\text{Var}})$.

We can then define new closure operators:

$$C_{MV}(\text{Q}) = C_T[\text{Q} \cup (\succsim_{\text{E}} \cap \succsim_{\text{Var}})] \tag{38}$$

$$C_{\text{FSD}}(\text{Q}) = C_T(\text{Q} \cup \succsim_{\text{FSD}}) \tag{39}$$

$$C_{\text{SSD}}(\text{Q}) = C_T(\text{Q} \cup \succsim_{\text{SSD}}) \tag{40}$$

and test these extended relations for some consistency axiom. See Heufer (2011) for a detailed analysis of stochastic dominance extensions.

5.2 *Further Possible Closure Operators*

Many other closure operators are possible. A very common assumption is that preferences are homothetic, and it is quite straightforward to impose this assumption on revealed preferences (see, for example, Knoblauch 1993).

Social preferences, preferences for altruism, or individual ideas about (distributive) justice have been analysed, among many others, by Karni and Safra (2002a,b, 2008) and Cox et al. (2008); experimental approaches include Andreoni and Miller (2002), Fisman et al. (2007), and Karni et al. (2008). Andreoni and Miller (2002) and Fisman et al. (2007) in particular use induced competitive budgets to examine subjects' preferences for giving money to other subjects in a generalised dictator game. Andreoni and Miller (2002) also included some non-monotonic budget sets to test preferences for monotonicity. The collected choices can then be used to derive the revealed preference relation over a subject's own payoff and the payoff of others. A possible extension could be based on the assumption that a subject at least weakly prefers (x_1, x_2) over (x_2, x_1) if $x_1 < x_2$, where x_1 is the own payoff and x_2 is the payoff to a different subject.

A further extension can be based on multiple-peaks of a preference as a generalisation of single-peakedness. Figure 3 shows an example of choices for which no single-peaked rationalising extension exists, but the choices can be rationalised by two-peaked preferences.

6 CONCLUSION

This paper provides a way to deal with choices from very general budget sets and shows how the revealed preference relation based on these choices can be extended using different closure operators which impose additional assumptions on the preferences of a decision maker. The paper derives several axioms which are possible to test and which provide the necessary and sufficient conditions for the existence of continuous complete extensions which satisfy several assumptions and rationalise the observed choices. The results are used for a nonparametric analysis of revealed preference data.

The paper is the first step towards an extensive analysis of what we can learn from observables about a decision maker's preferences under very general circumstances. Future work will focus on further applications of the methods presented here. One possible application is to reconsider the "falsifiable closure" introduced in Chambers et al. (2010) and to examine them under the assumption that we observe choices on varies families of budgets.

Another major step will be to use the methods to analyse a model where a decision maker's preference weakly depends on two relations, one of which is known a priori; the other relation is called the residual relation. The revealed preference relation and the a priori relation are then used to draw conclusions about the residual relation, based on the simple axiom that the intersection of the a priori and the residual relation is a subset of the preference. One example are risk preferences, where the expected value of lotteries is known a priori, and the residual relation is the DM's individual notion of riskiness. The residual relation captures the part of the DM's preference which can be used for interpersonal comparison of two different DMs.

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