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Universität Duisburg-Essen, Department of Economics
Universitätsstr. 12, 45117 Essen, Germany

Rheinisch-Westfälisches Institut für Wirtschaftsforschung (RWI)
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Philip Messow[1]

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Abstract

Synthetic Collateralized Debt Obligations (CDOs) were among the driving forces of the rapid growth of the market for credit derivatives in recent years. Possibly the most popular model beside the Gaussian copula for pricing CDO tranches is the Random-Factor-Loading-Model of Andersen and Sidenius (2005). We extend this model by allowing more than two regimes of default correlations. The model is calibrated to market spreads at times of financial distress and during calm periods. For both points in time the model correlation skews are similar to the steep skews observed in the market and lead to an improvement to the standard Random-Factor-Loading-Model.

JEL Classification: C58, G13

Keywords: Collateralized debt obligation; random-factor-loading; pricing; financial dependence; factor model; default risk; correlated defaults

February 2012

1 University of Dortmund (TU) and RGS Econ. – All correspondence to: Philip Messow, Fakultät Statistik, Technische Universität Dortmund, Vogelpothsweg 78, 44227 Dortmund, Germany, E-Mail: messow@statistik.tu-dortmund.de.

1 Introduction

The financial crisis that started in 2007 spread interest on the correct pricing of credit derivatives, like CDOs, as valuation of these derivatives became even more important within the last years.

The value of some CDO-tranche is the expected value of a function of the time of default of the underlying assets and therefore is affected by the correlation of the underlying assets (Brasch, 2004). It is common knowledge in the financial literature that defaults are correlated (Das *et al.*, 2002), but most practitioners still use the Gaussian copula approach introduced by Li (2000) for pricing CDO tranches even though the weaknesses of this approach are obvious (van der Voort, 2007). Due to the fact that this approach lacks of tail dependence (eg, Kole *et al.*, 2007), the reproducing of the observed correlation skew of the market spreads and especially the correct pricing of senior tranches is hardly possible. Andersen and Sidenius (2005) therefore extend the Gaussian copula approach by introducing the Random-Factor-Loading-Model with two correlation regimes (RFL(2)) accounting for the stylized fact that default correlations are higher in bear than in bull markets. This paper adds even more flexibility extending the basic RFL(2)-model to three correlation regimes (RFL(3)). We use a factor approach, introduced by Merton (1974), where default occurs if the firm's value of assets falls below a certain threshold. The reader is assumed to have some background in financial derivatives and so we omit background information on CDOs. For a short review we suggest section 2 of Glasserman and Suchintabandid (2007).

The paper is organized as follows. Section 2 introduces the general framework. An analytical expression of the RFL(3)-model is given in section 3. The model is calibrated to daily market spreads of the iTraxx 5yr observed during calm periods and in times of financial distress in section 4. Section 5 concludes and makes suggestions for future research.

2 Theoretical Background

We follow the general framework of Andersen and Sidenius (2005). A portfolio with I debitors, assuming all idiosyncratic default probabilities $p_i(t)$ at time t are known, exhibits a loss for the time interval $[0, T]$ of

$$L = \sum_{i=1}^{I} l_i \eta_{\tau_i \leq T} \tag{2.1}$$

with $\eta_{\tau_i \leq T}$ some indicator function taking the value 1, if creditor i defaults up to time T and 0 otherwise. l_i denotes the loss of debitor i. For a better understanding of the loss function variables $X_1, ..., X_I$ are introduced. Furthermore there exist threshold values $c_1, ..., c_I$ so that for all debitors $i = 1, ..., I$

$$\eta_{\tau_i \leq T} \equiv \eta_{X_i \leq c_i}. \tag{2.2}$$

By introducing a systematic factor Z with $Z = (Z_1, ..., Z_d)$ and $Z_i \sim IID(0; \sigma_Z)$, $X_1, ..., X_I$ and $l_1, ..., l_I$ are independent if conditioned on Z. Within this framework all debitors are affected by the same macroeconomic environment.

For the determination of the individual default probabilities, it is assumed that the random variable τ is distributed exponentially with constant intensity parameter λ. The time of default can be interpreted as the first jump of a poisson process.

$$Prob(\tau > t) = \exp(-\lambda \cdot t). \tag{2.3}$$

The default intensity λ is approximated by $\lambda = \frac{SP}{1-R}$, with SP the CDS credit spread and R the recovery rate (Li *et al.*, 2006).

The default-leg describes the expected payments of the protection purchaser and the premium-leg describes the expected payments of the protection seller.

The spread of the CDO has to be chosen in such way that the present values of both legs are equal. For simplicity reasons it is assumed that the defaults occur only at the observation points $j = 1, ..., J$, that is for a synthetic CDO written on iTraxx 5yr every three months. For further insights regarding the default timing see Finger (2005). The cumulative, percental loss of the tranche (A, B) is described by $L^*_{A,B}$. L^*_t represents the cumulative, percental loss of the portfolio and Δ_j describes the length of the period and $D(t_j)$ the discount factor for the period $[0; t_j]$, so that the tranche spread is given by

$$s_{A,B} = \frac{\sum_{j=1}^{J} D(t_j) \cdot (E(L^*_{A,B}(t_j)) - E(L^*_{A,B}(t_{j-1}))}{\sum_{j=1}^{J} \Delta_j \cdot D(t_j) \cdot (1 - E(L^*_{A,B}(t_j)))}. \tag{2.4}$$

The numerator describes the payments of the default-leg and the denominator the 'credit basis point value', the change in value resulting from a change in the spread of one base point. Using the representation 2.4, the following relationship holds

$$E(L^*_{A,B}(t)) = \frac{1}{B - A} \left[E \left(\max(L^*_t - A, 0) \right) - E \left(\max(L^*_t - B, 0) \right) \right]. \tag{2.5}$$

Because a tranche can be interpreted as an option on the portfolio losses, the expected loss, greater than A, can be described by a call-option with strike price A (Martin *et al.*, 2006):

$$E \left(\max(L^*_t - A, 0) \right) = 1 - A - \int_A^1 Prob(L^*_t \leqslant x) dx. \tag{2.6}$$

Until now, no distributional assumption of the portfolio losses has been made and the described framework holds for all models.

3 The Random Factor Loading Model with Three Regimes

The general Random-Factor-Loading-Model is described by the random variables $X_1, ..., X_I$ and l and the following relationship assuming dimension one for Z and homogeneity for all debitors

$$\left.\begin{aligned} X_i &= a(Z)\ Z + \nu\ \varepsilon_i + m \\ l &= l^{\max}(1 - R). \end{aligned}\right\} i = 1, ..., I \tag{3.7}$$

X_i has an expected value of 0 and a variance of 1 by choosing $\nu_i := \sqrt{1 - V(a(Z) \cdot Z)}$ and $m := -E(a(Z) \cdot Z)$. $a(Z) : \mathbb{R} \to \mathbb{R}^+$ is a deterministic function of the systematic factor Z.

Even though for the special case of a standard normal distributed factor Z and idiosyncratic factor ε_i, X_i is not standard normally distributed anymore, because the function $a(Z)$ is not constant.

Assuming a standard normal distribution, with $\phi(x)$ the pdf and $\Phi(x)$ the cdf, for the systematic factor Z with thresholds $\theta_i \in \mathbb{R}$, let

$$a_i = \begin{cases} \alpha & Z \leqslant \theta_1 \\ \beta & \theta_1 < Z \leqslant \theta_2 \\ \gamma & Z \geqslant \theta_2. \end{cases} \tag{3.8}$$

Depending on the realization of the systematic factor one of the three correlation regimes is switched on. To give an economic interpretation of the described parametrization it is possible to describe three different states of the economy, each state associated with a different correlation.

For the empirical verification the determination of m, ν and the unconditioned default probabilities are needed. The derivation is described briefly in the following with η some indicator function and in-depth in appendix $(A.1)$.

$$\begin{aligned} m &= -E(a(Z)\cdot Z) \\ &= -E(\alpha\cdot\eta_{Z\leqslant\theta_1}\cdot Z+\beta\cdot\eta_{\theta_1<Z\leqslant\theta_2}\cdot Z+\gamma\cdot\eta_{Z>\theta_2}\cdot Z) \\ &= (\alpha-\beta)\cdot\phi(\theta_1)+(\beta-\gamma)\cdot\phi(\theta_2). \end{aligned}$$

$$\begin{aligned} v :=& \sqrt{1-V(a(Z)\cdot Z)} \\ V(a(Z)\cdot Z) =& E\left(a(Z)^2\cdot Z^2\right)-\Bigg(\underbrace{E\left(a(Z)-Z\right)}_{-m}\Bigg)^2 \\ =& \alpha^2\cdot(\Phi(\theta_1)-\theta_1\cdot\phi(\theta_1)) \\ &+\beta^2\cdot(\eta_{\theta_2\geqslant\theta_1}\left(\Phi(\theta_2)-\Phi(\theta_1)\right)+\eta_{\theta_2\geqslant\theta_1}\left(\theta_1\cdot\phi(\theta_1)-\theta_2\cdot\phi(\theta_2)\right)) \\ &+\gamma^2\cdot(\theta_2\cdot\phi(\theta_2)+(1-\Phi(\theta_2))-m^2. \end{aligned}$$

Because the idiosyncratic default probabilities can be derived using the spread of the CDS, the following relationship of $c(t)$ and the unconditioned default probabilities hold with $\Phi_2(.)$ describing the cdf of the bivariate standardized normal distribution.

$$\begin{aligned} P(\tau\leqslant T) =& P(\alpha\cdot\eta_{Z\leqslant\theta_1}\cdot Z+\beta\cdot\eta_{\theta_1<Z\leqslant\theta_2}\cdot Z+\gamma\cdot\eta_{Z>\theta_2}\cdot Z+\varepsilon_i\cdot\nu+m\leqslant c) \\ =& \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}};\theta_1;\frac{\alpha}{\sqrt{\nu^2+\alpha^2}}\right) \\ &+\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};\theta_2;\frac{\beta}{\sqrt{\nu^2+\beta^2}}\right)-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};\theta_1;\frac{\beta}{\sqrt{\nu^2+\beta^2}}\right) \\ &+\Phi\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}}\right)-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}};\theta_2;\frac{\alpha}{\sqrt{\nu^2+\alpha^2}}\right). \end{aligned}$$

If m, ν and $c(t)$ are known, assuming $I \to \infty$, the unconditioned loss distribution with $\Omega(x) \equiv c - \nu \cdot \Phi^{-1}(x) - m$ can be derived by

$$\begin{aligned}
\lim_{I\to\infty} Prob\left(\frac{L}{l\cdot I} \geqslant x\right) =& Prob(p(Z) \geqslant x) = Prob(a(Z)\cdot Z \leqslant c - v \cdot \Phi^{-1}(x) - m) \\
=& \Phi\left(\min\left(\frac{\Omega(x)}{\alpha}\right), \theta_1\right) \\
&+ \eta_{\left(\frac{\Omega(x)}{\beta}\right)>\theta_1} \cdot \left(\Phi\left(\min\left(\frac{\Omega(x)}{\beta}\right), \theta_2\right) - \Phi(\theta_1)\right) \\
&+ \eta_{\left(\frac{\Omega(x)}{\gamma}\right)>\theta_2} \cdot \left(\Phi\left(\frac{\Omega(x)}{\gamma}\right) - \Phi(\theta_2)\right).
\end{aligned}$$

This loss function can be used to derive the expected loss of the tranches. Note that a total loss of an equity tranche with detachment point A occurs, if $\frac{A}{1-R}$ of the nominal value of the portfolio defaults. Using equation 2.6 one can derive the following relationship

$$\begin{aligned}
E\left(\max\left(L_t^* - A, 0\right)\right) =& 1 - A - \int_A^1 Prob(L_t^* \leqslant x)dx \\
=& 1 - A - \int_A^1 (1 - Prob(L_t^* > x))dx \\
=& 1 - A - (1 - A) + \int_A^1 Prob(L_t^* > x)dx \\
=& \int_A^1 \Phi\left(\min\left(\frac{\Omega(x)}{\alpha}\right), \theta_1\right)dx \\
&+ \int_A^1 \eta_{\left(\frac{\Omega(x)}{\beta}\right)>\theta_1} \cdot \left(\Phi\left(\min\left(\frac{\Omega(x)}{\beta}\right), \theta_2\right) - \Phi(\theta_1)\right)dx \\
&+ \int_A^1 \eta_{\left(\frac{\Omega(x)}{\gamma}\right)>\theta_2} \cdot \left(\Phi\left(\frac{\Omega(x)}{\gamma}\right) - \Phi(\theta_2)\right)dx.
\end{aligned}$$

Substituting $y = -\Phi^{-1}(x)$, so that $dx = -\phi(y)$ and $\Upsilon := -\Phi^{-1}(A)$, the following relationship, which is derived in detail in appendix $(A.1)$, is characterized.

$$
\begin{aligned}
y_\alpha &:= \frac{\theta_1 \cdot \alpha + m - c}{\nu} \\
y_\beta^1 &:= \frac{\theta_1 \cdot \beta + m - c}{\nu} \\
y_\beta^2 &:= \frac{\theta_2 \cdot \beta + m - c}{\nu} \\
y_\gamma &:= \frac{\theta_2 \cdot \gamma + m - c}{\nu}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\max\left(L_t^* - A, 0\right)\right) =& (1-R)\{\cdot \eta_{\Upsilon < y_\alpha} \cdot \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}}; \Upsilon; -\frac{\nu}{\sqrt{\nu^2+\alpha^2}}\right) \\
&+ \eta_{\Upsilon \geqslant y_\alpha} \cdot \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}}; y_\alpha; -\frac{\nu}{\sqrt{\nu^2+\alpha^2}}\right) \\
&+ \eta_{\Upsilon \geqslant y_\alpha} \cdot \Phi(\theta_1) \cdot \left(1 - \frac{A}{1-R} - \Phi(y_\alpha)\right) \qquad (3.10) \\
&+ \eta_{\Upsilon > y_\beta^1} \eta_{\Upsilon \leqslant y_\beta^2} \cdot \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}}; \Upsilon; -\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right) \\
&+ \eta_{\Upsilon > y_\beta^1} \eta_{\Upsilon \leqslant y_\beta^2} \cdot \left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}}; y_\beta^1; -\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\right) \\
&+ \eta_{\Upsilon > y_\beta^1} \eta_{\Upsilon \leqslant y_\beta^2} \cdot \left(-\Phi(\theta_1) \cdot \left(1 - \frac{A}{1-R} - \Phi(y_\beta^1)\right)\right) \\
&+ \eta_{\Upsilon > y_\beta^2} \cdot \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}}; y_\beta^2; -\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right) \\
&+ \eta_{\Upsilon > y_\beta^2} \cdot \left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}}; y_\beta^1; -\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\right) \\
&+ \eta_{\Upsilon > y_\beta^2} \cdot \left(-\Phi(\theta_1) \cdot \left(\Phi(y_\beta^2) - \Phi(y_\beta^1)\right)\right) \\
&+ \eta_{\Upsilon > y_\beta^2} \cdot \left(\Phi(\theta_2) - \Phi(\theta_1)\right) \cdot \left(1 - \frac{A}{1-R} - \Phi(y_\beta^2)\right) \\
&+ \eta_{\Upsilon > y_\gamma} \cdot \Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\gamma^2}}; \Upsilon; -\frac{\nu}{\sqrt{\nu^2+\gamma^2}}\right) \\
&+ \eta_{\Upsilon > y_\gamma} \cdot \left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\gamma^2}}; y_\gamma; -\frac{\nu}{\sqrt{\nu^2+\gamma^2}}\right)\right) \\
&+ \eta_{\Upsilon > y_\gamma} \cdot \left(-\Phi(\theta_2)\right)\left(1 - \frac{A}{1-R} - \Phi(y_\gamma)\right)\}.
\end{aligned}
$$

Using this analytical expression the expected tranche losses can be derived and used to determine the present values of the default- and premium-leg. Knowing these values, the model spreads can be calculated, which will be done in the next section.

4 Empirical verification

The main focus of this section is to check whether the RFL(3)-model is capable of producing the observed correlation skew in the market spreads and if this is done in such a way that one can assume that the third correlation regime has advantages over the parsimonious parametrization of the RFL(2)-model. We assume a flat yield curve from now on. Following Andersen and Sidenius (2005) a portfolio of 125 underlying assets, as the iTraxx 5yr, is a sufficiently big enough portfolio to apply the large portfolio limits of the last section. We assume a recovery rate of $R = 40\%$ because Altman and Kishore (1996) empirically investigated that the mean of the recovery rate lies around 40%. The models are calibrated to observed market spreads of the days 06/20/2007 and 09/16/2008. The first point in time is during a calm period and the second one is just after Lehman-Brothers defaulted, which describes one of the most volatile periods of financial history. In a second step the models are calibrated to two days simultaneously (06/20/2007-06/21/2007 & 09/16/2008-09/17/2008).

Table 1: Spreads of the iTraxx 5yr at 06/20/2007 and 09/16/2008 (in bp)

Date	$0\% - 3\%$	$3\% - 6\%$	$6\% - 9\%$	$9\% - 12\%$	$12\% - 22\%$
06/20/2007	723.31	47.44	12.54	5.70	2.27
09/16/2008	4864.10	685.93	414.69	239.84	114.09

Table 1 summarizes the market spreads of the 06/20/2007 and 09/16/2008. The spreads jumped in times of financial distress to a multiple of the observed spreads in calm periods.

The model spreads are a function of the vector $\vartheta = (\alpha, \beta, \gamma, \theta_1, \theta_2)'$, where α, β and γ are positive constants. The parameter vector ϑ is chosen so that $\min_{\vartheta} = \sum_{h=1}^{5} (\frac{s_h^{Model} - s_h^{Market}}{s_h^{Market}})^2$, where s_h^{Model} describes the spreads produced by the RFL(3)-model and s_h^{Market} describes the market spreads summarized in table 1. Given the default probability of the debitors, by using the CDS spreads, and some starting values $(\alpha_0; \beta_0; \gamma_0; \theta_{1_0}; \theta_{2_0})'$ the thresholds $c(t)$ can be derived. Table 2 summarizes the parametrization of the RFL(3)-model for both days.

Given the parametrization for the 06/20/2007, the α-regime can be inter-

Table 2: Parametrization of the RFL(3)-model for 06/20/2007 and 09/16/2008

Date	α	β	γ	θ_1	θ_2
06/20/2007	1.4027	0.4594	0.4012	−3.0712	−2.4912
09/16/2008	1.3815	1.1595	0.4876	−2.4192	−1.6993

preted as a 'disaster state' with high correlations of default. The probability of this state is $\Phi(-3.0712) = 0.107\%$. The correlation of the α-regime is roughly the same for both points in time. But the probability of obtaining this state at times of financial distress is about eight times higher ($\Phi(-2.4192) = 0.78\%$). As you can see in figure 1, the RFL(3)-model almost fits perfectly the observed skew of the base correlations for the time of financial distress, but the RFL(2)-model fit is just slightly worse. Fitting the base correlation skew perfectly for the calm period, the third correlation regime is an unambiguous improvement. Because the purpose of figure 1 was to show that both RFL-models can reproduce the basecorrelation skew with a better fit for the RFL(3)-model, table 3 and 4 summarize the deviations of the estimated model spreads to the market spreads for a more detailed view of the improvement due to the higher

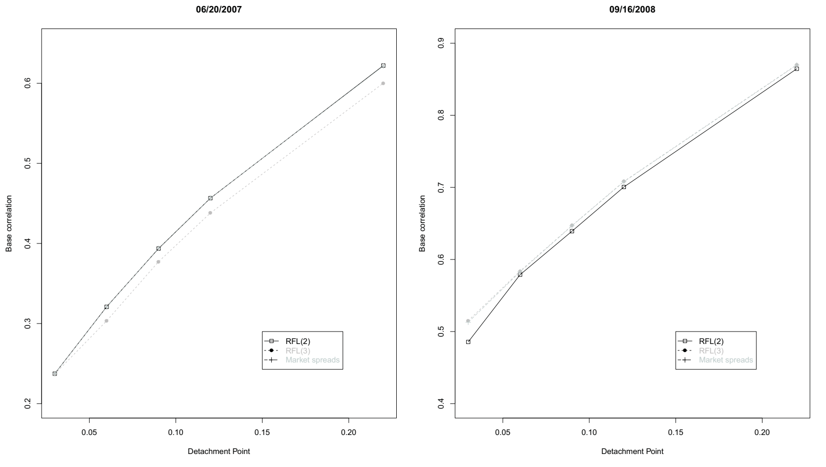

Figure 1: Comparison of the base correlations for the 06/20/2007 and 09/16/2008

parametrization. Compared to the RFL(2)-model the fit for the 06/20/2007 is much better since all tranches have a smaller percental deviation from market spreads. For the 09/16/2008 all tranches are priced as least as good as the RFL(2)-model, with a perfectly priced $12\% - 22\%$-tranche within both models. Overall, the introduction of a third correlation regime improves the results for both, the calm period and the period of financial distress.

Table 3: Deviation of the model from the market spreads for the 06/20/2007 (in %)

Model	$0\% - 3\%$	$3\% - 6\%$	$6\% - 9\%$	$9\% - 12\%$	$12\% - 22\%$
$Market$	723.31	47.44	12.55	5.70	2.27
$RFL(2) - Model$	0.17	15.86	16.31	1.21	0.44
$RFL(3) - Model$	0.00	0.00	0.00	0.00	0.00

Table 4: Deviation of the model from the market spreads for the 09/16/2008 (in %)

Model	$0\% - 3\%$	$3\% - 6\%$	$6\% - 9\%$	$9\% - 12\%$	$12\% - 22\%$
$Market$	4864.1	685.9	414.7	239.8	114.1
$RFL(2) - Model$	6.10	7.54	3.42	1.05	0.00
$RFL(3) - Model$	5.63	7.36	3.88	0.88	0.00

One could argue that the close to perfect fit of the RFL(3)-model is due to the fact that this model has five degrees of freedom by pricing five tranches. To show that the RFL(3)-model is a real improvement compared to the RFL(2)-model, we will calibrate the named models to the tranches of two days for a total of ten tranches to circumvent the described problem.

Looking at the percentage deviations from the market spreads in table 5, we find out that the fit for the first three tranches is much better within the RFL(3)-model than within the RFL(2)-model. The $9\% - 12\%$ & $12\% - 22\%$-tranches are fitted equally or slightly worse compared to the RFL(2)-model. Looking at the percentage deviations from the market spreads table 6 shows that the fit of the RFL(3)-model is much better for both days. Only the $6\% - 9\%$-tranche of the 09/16/2008 is fitted slightly worse while all other tranches are fitted much better.

Summing up the gains of introducing a third correlation regime are obvious. For trying to reproduce both the market spreads for one day and for several days, the gains of introducing a third correlation regime are worth its price for the additional parameters.

Table 5: Deviations of the RFL(2)- and RFL(3)-model from market spreads for the 06/20/2007 & 06/21/2007 (in percent)

Model	$0\% - 3\%$	$3\% - 6\%$	$6\% - 9\%$	$9\% - 12\%$	$12\% - 22\%$
06/20/2007($RFL3$)	14.89	20.16	12.34	3.51	0.84
06/20/2007($RFL2$)	0.10	0.92	3.60	3.71	2.86
06/21/2007($RFL3$)	12.30	24.56	11.62	1.69	2.05
06/21/2007($RFL2$)	0.23	1.21	4.26	3.71	3.20

5 Conclusion

'Collateralized Debt Obligations' revolutionized the market for credit derivatives during the last century. Because of the standardized trading of these derivatives there is a need for pricing the different tranches correctly. Within the framework of this paper a reasonable alternative to the Gaussian model which is the standard market model as well as to the well-known Random-Factor-Loading-Model by Andersen and Sidenius (2005) was introduced. The new model was calibrated to two points in time - one during calm financial markets and one during financial distress. For both points in time the empirical fit of the RFL(3)-model was much better than the fit of the Gaussian copula and considerably better than the fit of the RFL(2)-model.

Not only the lack of reproducing the empirically observed tail dependence enhanced the need for extensions to the Gaussian copula. The introduction of even more exotic derivatives, like CDO^2, increased this need dramatically. Within this paper we introduced such an extension.

A reasonable supplementation to the model described above would be to relaxe the assumption of constant recovery rates. Hamilton and Carty (1999) show that recovery rates in times of financial distress are much smaller than during

Table 6: Deviations of the RFL(2)- and RFL(3)-model from market spreads for the 09/16/2008 & 09/17/2008 (in percent)

Model	$0\% - 3\%$	$3\% - 6\%$	$6\% - 9\%$	$9\% - 12\%$	$12\% - 22\%$
09/16/2008($RFL2$)	4.00	9.63	1.11	0.89	0.97
09/16/2008($RFL3$)	0.02	0.79	1.21	0.83	0.00
09/17/2008($RFL2$)	9.97	6.20	5.86	3.31	0.98
09/17/2008($RFL3$)	1.07	0.58	0.54	1.40	0.00

calm periods. Introducing stochastic recovery rates into the RFL-models by making R a function of the systematic factor Z would be one way of dealing with this stylized fact.

Another interesting part of future research would be to look how well the different models do if they are calibrated to even more days in time than just two. By doing so one could find out how sensitive these models react to parameter changes.

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A Appendix

A.1 Derivation of the large portfolio limit of the RFL(3)-model

The following relationship of m and ν results in the RFL(3)-model with $\theta_2 \geqslant \theta_1$

$$
\begin{aligned}
m =& - E(a(Z) \cdot Z) = -E(\alpha \cdot \eta_{Z \leqslant \theta_1} \cdot Z + \beta \cdot \eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z + \gamma \cdot \eta_{Z > \theta_2} \cdot Z) \\
=& - (\alpha \cdot (-\phi(\theta_1)) + \beta \cdot \eta_{\theta_2 \geqslant \theta_1} \cdot (\phi(\theta_1) - \phi(\theta_2)) + \gamma \cdot \phi(\theta_2)) \\
=& - (-\alpha \cdot \phi(\theta_1) + \beta \cdot (\phi(\theta_1) - \phi(\theta_2)) + \gamma \cdot \phi(\theta_2)) \\
=& - (-\alpha \cdot \phi(\theta_1) + \beta \cdot \phi(\theta_1) - \beta \cdot \phi(\theta_2) + \gamma \cdot \phi(\theta_2)) \\
=& (\alpha - \beta) \cdot \phi(\theta_1) + (\beta - \gamma) \cdot \phi(\theta_2)
\end{aligned}
$$

and

$$
\begin{aligned}
v_i :=& \sqrt{1 - V(a_i(Z) \cdot Z)} \\
V(a_i(Z) \cdot Z) =& E\left(a_i(Z)^2 \cdot Z^2\right) - \Bigg(\underbrace{E\left(a_i(Z) - Z\right)}_{-m_i}\Bigg)^2 \\
=& E(\alpha^2 \cdot \eta_{Z \leqslant \theta_1} \cdot Z^2 + \beta^2 \cdot \eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z^2 + \gamma^2 \cdot \eta_{Z > \theta_2} \cdot Z^2) \\
=& \alpha^2 \cdot E(\eta_{Z \leqslant \theta_1} \cdot Z^2) + \beta^2 \cdot E(\eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z^2) + \gamma^2 \cdot E(\eta_{Z > \theta_2} \cdot Z^2) \\
=& \alpha^2 \cdot (\Phi(\theta_1) - \theta_1 \cdot \phi(\theta_1)) \\
& + \beta^2 \left(\eta_{\theta_2 \geqslant \theta_1} \left(\Phi(\theta_2) - \Phi(\theta_1)\right) + \eta_{\theta_2 \geqslant \theta_1} \left(\theta_1 \cdot \phi(\theta_1) - \theta_2 \cdot \phi(\theta_2)\right)\right) \\
& + \gamma^2(\theta_2 \cdot \phi(\theta_2) + (1 - \Phi(\theta_2)) - m^2.
\end{aligned}
$$

The unconditional probability of default is described as follows

$$
\begin{aligned}
Prob(\tau \leqslant T) =& Prob(\alpha \cdot \eta_{Z \leqslant \theta_1} \cdot Z + \beta \cdot \eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z + \gamma \cdot \eta_{Z > \theta_2} \cdot Z + \varepsilon_i \cdot \nu + m \leqslant c) \\
=& E\left(Prob\left(\varepsilon_i \leqslant \frac{c - \alpha \cdot \eta_{Z \leqslant \theta_1} \cdot Z - \beta \cdot \eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z - \gamma \cdot \eta_{Z > \theta_2} \cdot Z - m}{\nu} | Z \right) \right) \\
=& E\left(\Phi\left(\frac{c - \alpha \cdot \eta_{Z \leqslant \theta_1} \cdot Z - \beta \cdot \eta_{\theta_1 < Z \leqslant \theta_2} \cdot Z - \gamma \cdot \eta_{Z > \theta_2} \cdot Z - m}{\nu} \right) \right) \\
=& \int\limits_{-\infty}^{\theta_1} \Phi\left(\frac{c - \alpha \cdot Z - m}{\nu} \right) \cdot \phi(z) dz + \int\limits_{\theta_1}^{\theta_2} \Phi\left(\frac{c - \beta \cdot Z - m}{\nu} \right) \cdot \phi(z) dz \\
& + \int\limits_{\theta_2}^{\infty} \Phi\left(\frac{c - \gamma \cdot Z - m}{\nu} \right) \cdot \phi(z) dz \\
=& \Phi_2\left(\frac{c - m}{\sqrt{\nu^2 + \alpha^2}}; \theta_1; \frac{\alpha}{\sqrt{\nu^2 + \alpha^2}} \right) \\
& + \Phi_2\left(\frac{c - m}{\sqrt{\nu^2 + \beta^2}}; \theta_2; \frac{\beta}{\sqrt{\nu^2 + \beta^2}} \right) - \Phi_2\left(\frac{c - m}{\sqrt{\nu^2 + \beta^2}}; \theta_1; \frac{\beta}{\sqrt{\nu^2 + \beta^2}} \right) \\
& + \Phi\left(\frac{c - m}{\sqrt{\nu^2 + \gamma^2}} \right) - \Phi_2\left(\frac{c - m}{\sqrt{\nu^2 + \gamma^2}}; \theta_2; \frac{\gamma}{\sqrt{\nu^2 + \gamma^2}} \right).
\end{aligned}
$$

Using the parameters from above one can derive the unconditional loss distribution.

$$
\begin{aligned}
\lim_{n\to\infty} Prob\left(\frac{L^*}{n \cdot l} \geqslant x\right) =& Prob(p(Z) \geqslant x) \\
=& Prob(a(Z) \cdot Z \leqslant c - v \cdot \Phi^{-1}(x) - m) \\
=& Prob(\alpha \cdot Z \leqslant \Omega(x), Z \geqslant \theta_1) + Prob(\beta \cdot Z \leqslant \Omega(x), \theta_1 < Z \leqslant \theta_2) \\
& + Prob(\gamma \cdot Z \leqslant \Omega(x), Z > \theta_2) \\
=& Prob\left(Z \leqslant \left(\min\left(\frac{\Omega(x)}{\alpha}\right), \theta_1\right)\right) \\
& + \eta_{\left(\frac{\Omega(x)}{\beta}\right) > \theta_1} \cdot \left(\Phi\left(\min\left(\frac{\Omega(x)}{\beta}\right), \theta_2\right) - \Phi(\theta_1)\right) \\
& + \eta_{\left(\frac{\Omega(x)}{\gamma}\right) > \theta_2} \cdot \left(\Phi\left(\frac{\Omega(x)}{\gamma}\right) - \Phi(\theta_2)\right) \\
=& \Phi\left(\min\left(\frac{\Omega(x)}{\alpha}\right), \theta_1\right) + \eta_{\left(\frac{\Omega(x)}{\beta}\right) > \theta_1} \cdot \left(\Phi\left(\min\left(\frac{\Omega(x)}{\beta}\right), \theta_2\right) - \Phi(\theta_1)\right) \\
& + \eta_{\left(\frac{\Omega(x)}{\gamma}\right) > \theta_2} \cdot \left(\Phi\left(\frac{\Omega(x)}{\gamma}\right) - \Phi(\theta_2)\right)
\end{aligned}
$$

Using 2.6 one can prove 3.10. It is necessary that

$$
\begin{aligned}
y_\alpha &:= \frac{\theta_1 \cdot \alpha + m - c}{\nu} \\
y_\beta^1 &:= \frac{\theta_1 \cdot \beta + m - c}{\nu} \\
y_\beta^2 &:= \frac{\theta_2 \cdot \beta + m - c}{\nu} \\
y_\gamma &:= \frac{\theta_2 \cdot \gamma + m - c}{\nu}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\max\left(L^{*}-A,0\right)\right) =& 1-A-\int_{A}^{1} Prob(L_t^{*}\leqslant x)dx \\
=& 1-A-\int_{A}^{1}\left(1-Prob(L_t^{*}>x)\right)dx \\
=& 1-A-(1-A)+\int_{A}^{1} Prob(L_t^{*}>x)dx \\
=& \underbrace{\int_{A}^{1}\Phi\left(\min\left(\frac{\Omega(x)}{\alpha}\right),\theta_1\right)dx}_{I_1} \\
&+\underbrace{\int_{A}^{1}\eta_{\left(\frac{\Omega(x)}{\beta}\right)>\theta_1}\cdot\left(\Phi\left(\min\left(\frac{\Omega(x)}{\beta}\right),\theta_2\right)-\Phi(\theta_1)\right)dx}_{I_2} \\
&+\underbrace{\int_{A}^{1}\eta_{\left(\frac{\Omega(x)}{\gamma}\right)>\theta_2}\cdot\left(\Phi\left(\frac{\Omega(x)}{\gamma}\right)-\Phi(\theta_2)\right)dx}_{I_3}
\end{aligned}
$$

Using the substitution $y=-\Phi^{-1}(x)$, $dx=-\phi(y)$ and $\Upsilon:=-\Phi^{-1}(A)$ the results of the derivations of the integrals I_1, I_2 and I_3 are described in the following. This yields for I_1

$$
\int_{-\infty}^{\Upsilon}\Phi\left(\min\left(\frac{c+\nu\cdot y-m}{\alpha},\theta_1\right)\right)\cdot\phi(y)dy.
$$

$\frac{c+\nu\cdot y-m}{\alpha}$ is smaller than θ_1 if and only if $y\leqslant y_a$. So we have to discriminate between the two cases $\Upsilon<y_a$ and $\Upsilon\geqslant y_a$. This yields for $\Upsilon<y_a$

$$
\int_{-\infty}^{\Upsilon}\Phi\left(\left(\frac{c+\nu\cdot y-m}{\alpha}\right)\right)\cdot\phi(y)dy=\Phi_2\left(\frac{c-m}{\sqrt{\alpha^2+\nu^2}};\Upsilon;-\frac{\nu}{\sqrt{\alpha^2+\nu^2}}\right)
$$

and for $\Upsilon\geqslant y_a$

$$
\begin{aligned}
&\int_{-\infty}^{y_\alpha}\Phi\left(\left(\frac{c+\nu\cdot y-m}{\alpha}\right)\right)\cdot\phi(y)dy+\int_{y_\alpha}^{\Upsilon}\Phi(\theta_1)\cdot\phi(y)dy \\
&=\Phi_2\left(\frac{c-m}{\sqrt{\alpha^2+\nu^2}};y_\alpha;-\frac{\nu}{\sqrt{\alpha^2+\nu^2}}\right)+\Phi(\theta_1)\cdot\left(1-A-\Phi(y_\alpha)\right).
\end{aligned}
$$

The following result holds for I_2

$$\int_{-\infty}^{\Upsilon} \eta_{\frac{c+\nu y-m}{\beta} \geqslant \theta_1} \cdot \Phi \left(\min \left(\frac{c+\nu \cdot y - m}{\beta}, \theta_2 \right) - \Phi(\theta_1) \right) \cdot \phi(y) dy$$
$$= \int_{-\infty}^{\Upsilon} \eta_{y \geqslant y_\beta^1} \cdot \Phi \left(\min \left(\frac{c+\nu \cdot y - m}{\beta}, \theta_2 \right) - \Phi(\theta_1) \right) \cdot \phi(y) dy.$$

If $y < y_\beta^1$ the indicator function is always 0, and the integral can be transformed as follows given $\Upsilon \geqslant y_\beta^1$

$$\int_{y_\beta^1}^{\Upsilon} \Phi \left(\min \left(\frac{c+\nu \cdot y - m}{\beta}, \theta_2 \right) - \Phi(\theta_1) \right) \cdot \phi(y) dy.$$

A distinction of cases is necessary for the derivation of I_1, $\Upsilon < y_\beta^2$ and $\Upsilon \geqslant y_\beta^2$, because $\frac{c+\nu \cdot y - m}{\beta}$ is smaller than θ_2 if and only if $y \leqslant y_\beta^2$. This yields for $\Upsilon < y_\beta^2$

$$\int_{y_\beta^1}^{\Upsilon} (\Phi \left(\frac{c+\nu \cdot y - m}{\beta} \right) - \Phi(\theta_1)) \cdot \phi(y) dy$$
$$= \int_{-\infty}^{\Upsilon} \Phi \left(\frac{c+\nu \cdot y - m}{\beta} \right) \cdot \phi(y) dy - \int_{-\infty}^{y_\beta^1} \Phi \left(\frac{c+\nu \cdot y - m}{\beta} \right) \cdot \phi(y) dy - \Phi(\theta_1) \cdot \int_{y_\beta^1}^{\Upsilon} \phi(y) dy$$
$$= \Phi_2 \left(\frac{c-m}{\sqrt{\beta^2+\nu^2}}; \Upsilon; -\frac{\nu}{\sqrt{\beta^2+\nu^2}} \right) - \Phi_2 \left(\frac{c-m}{\sqrt{\beta^2+\nu^2}}; y_\beta^1; -\frac{\nu}{\sqrt{\beta^2+\nu^2}} \right)$$
$$- \Phi(\theta_1) \cdot \left(1 - A - \Phi(y_\beta^1) \right)$$

and for $\Upsilon \geqslant y_\beta^2$

$$\begin{aligned}
&\int_{y_\beta^1}^{y_\beta^2} \left(\Phi\left(\frac{c+\nu\cdot y-m}{\beta}\right) - \Phi(\theta_1)\right)\cdot\phi(y)dy + \int_{y_\beta^2}^{\Upsilon} \left(\Phi(\theta_2)-\Phi(\theta_1)\right)\cdot\phi(y)dy \\
=&\int_{y_\beta^1}^{y_\beta^2} \Phi\left(\frac{c+\nu\cdot y-m}{\beta}\right)\cdot\phi(y)dy - \Phi(\theta_1)\cdot\int_{y_\beta^1}^{y_\beta^2}\phi(y)dy + \left(\Phi(\theta_2)-\Phi(\theta_1)\right)\int_{y_\beta^2}^{\Upsilon}\phi(y)dy \\
=&\int_{-\infty}^{y_\beta^2} \Phi\left(\frac{c+\nu\cdot y-m}{\beta}\right)\cdot\phi(y)dy - \int_{-\infty}^{y_\beta^1} \Phi\left(\frac{c+\nu\cdot y-m}{\beta}\right)\cdot\phi(y)dy \\
&-\Phi(\theta_1)\cdot\left(\Phi(y_\beta^2)-\Phi(y_\beta^1)\right) + \left(\Phi(\theta_2)-\Phi(\theta_1)\right)\cdot\left(1-A-\Phi(y_\beta^2)\right) \\
=&\Phi_2\left(\frac{c-m}{\sqrt{\beta^2+\nu^2}}; y_\beta^2; -\frac{\nu}{\sqrt{\beta^2+\nu^2}}\right) - \Phi_2\left(\frac{c-m}{\sqrt{\beta^2+\nu^2}}; y_\beta^1; -\frac{\nu}{\sqrt{\beta^2+\nu^2}}\right) \\
&-\Phi(\theta_1)\cdot\left(\Phi(y_\beta^2)-\Phi(y_\beta^1)\right) + \left(\Phi(\theta_2)-\Phi(\theta_1)\right)\cdot\left(1-A-\Phi(y_\beta^2)\right).
\end{aligned}$$

In a last step one has to derive the integral I_3. Given that $\Upsilon \geqslant y_\gamma$ the following relationship holds:

$$\int_{-\infty}^{\Upsilon} \eta_{\frac{c+\nu\cdot y-m}{\gamma}\geqslant\theta_2}\cdot\left(\Phi\left(\frac{c+\nu\cdot y-m}{\gamma}\right)-\Phi(\theta_2)\right)\cdot\phi(y)dy.$$

Because $\frac{c+\nu\cdot y-m}{\gamma} \geqslant \theta_2$ holds if and only if $y \geqslant y_\gamma$, I_3 can be transformed to

$$\int_{y_\gamma}^{\Upsilon} \left(\Phi\left(\frac{c+\nu\cdot y-m}{\gamma}\right)-\Phi(\theta_2)\right)\cdot\phi(y)dy.$$

This integral can be derived without using a distinction of cases

$$
\begin{aligned}
&\int\limits_{y_\gamma}^{\Upsilon} \left(\Phi\left(\frac{c+\nu\cdot y-m}{\gamma} \right) - \Phi(\theta_2) \right) \cdot \phi(y) dy \\
=&\int\limits_{y_\gamma}^{\Upsilon} \Phi\left(\frac{c-\nu\cdot y-m}{\gamma} \right) \cdot \phi(y) dy - \int\limits_{y_\gamma}^{\Upsilon} \Phi\left(\theta_2\right) \cdot \phi(y) dy \\
=&\Phi_2\left(\frac{c-m}{\sqrt{\gamma^2+\nu^2}}; \Upsilon; -\frac{\nu}{\sqrt{\gamma^2+\nu^2}} \right) - \Phi_2\left(\frac{c-m}{\sqrt{\gamma^2+\nu^2}}; y_\gamma; -\frac{\nu}{\sqrt{\gamma^2+\nu^2}} \right) \\
&- \Phi(\theta_2) \cdot (1 - A - \Phi(y_\gamma)) \,.
\end{aligned}
$$

Adding all integrals up, results in equation 3.10

$$
\begin{aligned}
E\left(\max\left(L^{*}-A,0\right)\right) =& (1-R)\{\cdot\eta_{\Upsilon<y_\alpha}\cdot\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}};\Upsilon;-\frac{\nu}{\sqrt{\nu^2+\alpha^2}}\right)\\
&+\eta_{\Upsilon\geqslant y_\alpha}\cdot\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\alpha^2}};y_\alpha;-\frac{\nu}{\sqrt{\nu^2+\alpha^2}}\right)\\
&+\eta_{\Upsilon\geqslant y_\alpha}\cdot\Phi(\theta_1)\cdot\left(1-\frac{A}{1-R}-\Phi(y_\alpha)\right)\\
&+\eta_{\Upsilon>y_\beta^1}\eta_{\Upsilon\leqslant y_\beta^2}\cdot\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};\Upsilon;-\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\\
&+\eta_{\Upsilon>y_\beta^1}\eta_{\Upsilon\leqslant y_\beta^2}\cdot\left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};y_\beta^1;-\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\right)\\
&+\eta_{\Upsilon>y_\beta^1}\eta_{\Upsilon\leqslant y_\beta^2}\cdot\left(-\Phi(\theta_1)\cdot\left(1-\frac{A}{1-R}-\Phi(y_\beta^1)\right)\right)\\
&+\eta_{\Upsilon>y_\beta^2}\cdot\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};y_\beta^2;-\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\\
&+\eta_{\Upsilon>y_\beta^2}\cdot\left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\beta^2}};y_\beta^1;-\frac{\nu}{\sqrt{\nu^2+\beta^2}}\right)\right)\\
&+\eta_{\Upsilon>y_\beta^2}\cdot\left(-\Phi(\theta_1)\cdot\left(\Phi(y_\beta^2)-\Phi(y_\beta^1)\right)\right)\\
&+\eta_{\Upsilon>y_\beta^2}\cdot\left(\Phi(\theta_2)-\Phi(\theta_1)\right)\cdot\left(1-\frac{A}{1-R}-\Phi(y_\beta^2)\right)\\
&+\eta_{\Upsilon>y_\gamma}\cdot\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\gamma^2}};\Upsilon;-\frac{\nu}{\sqrt{\nu^2+\gamma^2}}\right)\\
&+\eta_{\Upsilon>y_\gamma}\cdot\left(-\Phi_2\left(\frac{c-m}{\sqrt{\nu^2+\gamma^2}};y_\gamma;-\frac{\nu}{\sqrt{\nu^2+\gamma^2}}\right)\right)\\
&+\eta_{\Upsilon>y_\gamma}\cdot\left(-\Phi(\theta_2)\right)\left(1-\frac{A}{1-R}-\Phi(y_\gamma)\right)\}.
\end{aligned}
$$