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Julia Belau

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Universität Duisburg-Essen, Department of Economics
Universitätsstr. 12, 45117 Essen, Germany

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Julia Belau[1]

# Outside Options in Probabilistic Coalition Situations

## Abstract

*In this paper, I introduce an extension of (TU) games with a coalition structure. Taking a situation where all coalitions are already established is not reasonable in order to forecast the reality; there is not only one possible coalition, there are several. I consider situations where coalitions are not established yet and take into account the likelihood of each possible coalition. This leads to a generalized, probabilistic setting for coalition structures. Probabilistic versions of known axioms are introduced as well as new probabilistic axioms. Generalizations of both the outside-option-sensitive chi-value (Casajus, Soc Choice Welf 32, 1-13, 2009) and its outside-option-insensitive pendant, the component restricted Shapley value (Aumann and Drèze, Int. J. Game Theory 3, 217-237, 1974), are defined and axiomatic characterizations are given.*

*JEL Classification: C71*

*Keywords: TU game; coalition structure; outside option*

*January 2011*

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*1 Ruhr Graduate School in Economics and TU Dortmund.* *– I wish to thank PD Dr. André Casajus for his advice and helpful discussions. I also thank Prof. Dr. Wolfgang Leininger and Enrico Paolo Bugarin for their helpful comments.* *– All correspondence to Julia Belau, c/o RWI, Hohenzollern-str. 1-3, 45128 Essen, Germany, E-Mail: julia.belau@rgs-econ.de.*

# 1 Introduction

Most research on coalition structures assumes that some certain social or economic structure is already present, i.e. the individuals have already formed certain coalitions. But is it reasonable to take such situations as completely given? Mostly, there is not only one possible structure, there are several. Take for example elections: at the time of election, it is not known which parties will build coalitions, but there are some beliefs or assumptions about all possible coalitions. For forecasting, it is not reasonable to take situations as given, one should include uncertainty. From a more general point of view, the framework can be modelled by starting at some point *before* coalitions are formed.

I consider all possible structures and their likelihood, while it would also be possible to think of structures as the result of independent relations. Gómez et al. (2008), who extend the setting of networks, i.e. bilateral relations between the individuals, explain why "the importance of removing the independence assumption should not be underestimated" by features like incompatibilities: the presence of a certain relation between persons, enterprises, or political parties (a bilateral relation or also a formed coalition) can exclude the possibility of a relation between one of them and a third actor (Gómez et al., 2008, page 540). Take political parties: A coailition between two parties might exclude the possibility of a coalition between them and a third one due insuperable conflicts between this third party and one of the first two. Situations where structures are already present are called deterministic and, referring to the likelihood assumption, the extended situations probabilistic.

In this paper I extend the deterministic setting of coalition structures to a probabilistic one and define and study a probabilistic extension of an outside-option-sensitive solution concept and its features. Why is sensitivity to outside options, which is for example taking into account a bargaining position of a certain player against other players, an important and reasonable feature? Pfau (2008) shows in his experimental work about social interchange that outside options significantly affect negotiation. Any formed coalition between individuals "only describes one particular consideration". The result of negotiation between the individuals will be "decisively influenced by the other alliances which each one might alternatively have entered". "Even if [...] one particular alliance is actually formed, the others are present in virtual existence: Although they have not materialized, they have contributed essentially to shaping and determining the actual reality." (von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior, Princeton University Press, 1944, p. 36)

Casajus (2009a) and Wiese (2007) both introduce outside-option-sensitive solution concepts for coalition situations. I focus on the $\chi$-value (Casajus, 2009a): "The Wiese value has some drawbacks. Most notably it lacks a 'nice' axiomatization. In essence, there is a non-intuitive ad-hoc specification of the payoffs [...]." (Casajus, 2009a, page 50)

The following example shows why an outside-option-sensitive value is reasonable for distributing wealth on the individuals and how the probabilistic framework can be used to forecast the outputs of all individuals given some certain beliefs.

Consider a gloves game (Shapley and Shubik, 1969) which is a well-known market game in cooperative game theory and is suitable to model negotiation situations with asymmetric distribution of power (Pfau, 2008, page 8). Take such a game with six players, two of whom with left gloves, which we denote by $l_1, l_2$, and the remaining four with right gloves, denoted by $r_1, r_2, r_3, r_4$. The worth of a coalition is the number of matching glove-pairs. Suppose that four players form two matching pairs leaving the remaining two players (right-glove holders) unattached. The coalitions could be described by $\mathcal{P} = \{\{l_1, r_1\}, \{l_2, r_2\}, \{r_3\}, \{r_4\}\}$. How should the worth (1 per matching pair) be distributed among the players?

Casajus (2009a) compares the outcome for this situation of his outside-option-sensitive $\chi$-value with two well known solution concepts: the Shapley value (Shapley, 1953) and the component restricted Shapley value (Aumann and Drèze, 1974). Find the values of this example in Table 1.

| glove holder | $\chi$- value | AD-value | Shapley value |
|---|---|---|---|
| $l_1$, $l_2$ | 0.8 | 0.5 | 0.7333 |
| $r_1$, $r_2$ | 0.2 | 0.5 | 0.1333 |
| $r_3$, $r_4$ | 0 | 0 | 0.1333 |

Table 1: Payoffs for the gloves game

The Shapley value does not take into account any coalition structure, all right-glove holders obtain the same payoff even though two of them are not in a coalition that creates worth. The AD-value (depending on a player's own coalition only) splits the worth of 1 equally within matching pairs, even though the left-glove holders have a better bargaining position due to outside options: a left-glove holder could argue that he could form a coalition with another right-glove holder instead of the current one and create the same worth. The $\chi$-value takes into account outside options of the players as well as the coalition structure.

Let us now consider the situation where coalitions are not yet established but there are some (common) "beliefs" about all possible coalitions, for example due to statistical forecasts or some special features. The situation can be considered as a probabilistic coalition situation, that is, one considers a probability distribution over all possible coalitions.
Let $r_1$ and $r_3$ be more attractive to the left-glove holder $l_1$ while $l_2$ prefers $r_2$ and $r_4$ (for example due to slightly different colors). Further, the right-glove holders *suppose* $l_1$ to prefer $r_1$ over $r_3$ (due to some different features) while they *suppose* $l_2$ not to distinguish between $r_2$ and $r_4$. Taking into account these beliefs we are left to consider only the following coalition

structures:

$$\mathcal{P}_1 = \{\{l_1, r_1\}, \{l_2, r_2\}, \{r_3\}, \{r_4\}\},\ \mathcal{P}_2 = \{\{l_1, r_1\}, \{l_2, r_4\}, \{r_3\}, \{r_2\}\}$$
$$\mathcal{P}_3 = \{\{l_1, r_3\}, \{l_2, r_2\}, \{r_1\}, \{r_4\}\},\ \mathcal{P}_4 = \{\{l_1, r_3\}, \{l_2, r_4\}, \{r_1\}, \{r_2\}\}$$

Let the beliefs of the right-glove holders lead to the following probabilities:

$$p(\mathcal{P}_1) = p(\mathcal{P}_2) = 0.3,\ p(\mathcal{P}_3) = p(\mathcal{P}_4) = 0.2$$

where $p(\mathcal{P}_i)$ is the probability that the coalition structure $\mathcal{P}_i$ occurs.

For this situation, the values for the probabilistic $\chi$-value and the probabilistic AD-value can be derived as given in Table 2.

| glove holder | probabilistic $\chi$- value | probabilistic AD-value |
|---|---|---|
| $l_1$, $l_2$ | 0.8 | 0.5 |
| $r_1$ | 0.12 | 0.3 |
| $r_2$ | 0.1 | 0.25 |
| $r_3$ | 0.08 | 0.2 |
| $r_4$ | 0.1 | 0.25 |

Table 2: Expected ex ante payoffs for the gloves game

The values can be interpreted as the expected ex ante payoffs of the right-glove holders (the left-glove holders will obtain their share of the worth for sure). It turns out that the values are the same as the probability-weighted sum of the deterministic values. That is not surprising, since this seems to be a reasonable definition for probabilistic values.

Nevertheless, it is interesting to consider the more general setting and to find more general axioms than the deterministic ones to characterize values while still having the incentives of the deterministic values as sensitivity to outside options.

The paper is organized as follows: The next chapter gives notations of deterministic coalition structures and the $\chi$-value with its features. Chapter three introduces the probabilistic setting as well as the probabilistic $\chi$-value and gives a first "straight forward"-characterization. In chapter four, general probabilistic features are introduced, a characterization of the probabilistic $\chi$-value and a link it to the first characterization are given. Chapter five gives a characterization of the "outside-option-insensitive pendant" of the probabilistic $\chi - value$, an extension of the component restricted Shapley value (Aumann and Drèze, 1974).

# 2 Deterministic Coalition Situations

Let $N = \{1, 2, ..., n\}$ be a non-empty, finite set of players. A (TU) game is a pair $(N, v)$, where $v$, the coalition function, is a real function defined on $2^N$, the set of all subsets of $N$, satisfying $v(\emptyset) = 0$. For all $K \subseteq N$, $v(K)$ represents the economic possibilities of players in $K$, i.e. the worth of the coalition $K$. For $T \subseteq N, T \neq \emptyset$, the game $(N, u_T)$, $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise is called a unanimity game.
A solution concept is a function $\varphi$ that assigns payoff vectors for all games $(N, v)$, i.e. $\varphi(N, v) \in \mathbb{R}^N$. A very used solution concept for (TU) games is the Shapley value $Sh$ (Shapley, 1953), for any player $i \in N$ given by

$$Sh_i(N,v) := \sum_{K \subseteq N \setminus \{i\}} \frac{k!(n-1-k)!}{n!} [v(K \cup \{i\}) - v(K)], \tag{2.1}$$
$$\text{where } k = |K|, n = |N|$$

The Shapley value assigns to every player the average marginal contribution over all orders, where the marginal contribution is given by $v(K \cup \{i\}) - v(K)$.
A partition $\mathcal{P} \subseteq 2^N$ is called a *coalition structure (CS)* for $(N, v)$, where $\mathcal{P}(i)$ denotes the component containing player $i$ (the set of all players that are in the same cell as player $i$). A CS-game or *coalition situation* $(N, v, \mathcal{P})$ is a (TU) game together with a coalition structure. A CS-value $\varphi$ assigns payoff vectors for all CS-games. For any $K \subseteq N$ I define $\varphi_K(N, v, \mathcal{P}) := \sum_{i \in K} \varphi_i(N, v, \mathcal{P})$.

Now I give definitions and motivations of features for CS-values (based on Casajus, 2009a). These features are called CS-axioms.
**Additivity (A)** A CS-value $\varphi$ satisfies **A**, if for any coalition functions $v, w$: $\varphi(N, v+w, \mathcal{P}) = \varphi(N, v, \mathcal{P}) + \varphi(N, w, \mathcal{P})$.
**A** is a standard axiom.[1] It is satisfied by most of the solution concepts referred to in the introduction.
**Component Efficieny (CE)** A CS-value $\varphi$ satisfies **CE**, if $\varphi_{\mathcal{P}(i)}(N, v, \mathcal{P}) = v(\mathcal{P}(i))$ for all $i \in N$.
The motivation of this axiom is that players within a component, seen as a productive unit, cooperate to create the component's worth. In contrast there is the *Efficiency axiom* **E**: $\varphi_N(N, v, \mathcal{P}) = v(N)$.
**Component Restricted Symmetry (CS)** Players $i, j \in N$ are called symmetric in $(N, v)$ if $v(K \cup \{i\}) = v(K \cup \{j\}) \forall K \subseteq N \setminus \{i, j\}$. A CS-value $\varphi$ satisfies **CS**, if $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$ for all symmetric Players $i, j \in N$ and $j \in \mathcal{P}(i)$.
Symmetric players have the same productivity. CS-values should provide the same payoff for players with equal productivity that are in the same component, since there is nothing like an inner structure which could be reasonable for a different treatment of these players. **CS**

[1] For a motivation of **A** see Roth, A.E.(1977): The Shapley value as a von Neumann-Morgenstern utility. Econometrica 45, p. 657-664

is a relaxation of the *Symmetry axiom* **S**: $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$ for all symmetric players in $(N, v)$.

**Grand Coalition Null Player (GN)** A player $i \in N$ is called a Null player in $(N, v)$ if $v(K \cup \{i\}) = v(K)\, \forall K \subseteq N \setminus \{i\}$. A CS-value $\varphi$ satisfies **GN**, if $\varphi_i(N, v, \{N\}) = 0$ for all Null players $i \in N$.

This axiom is a relaxation of the *Null player axiom* **N**: $\varphi_i(N, v, \mathcal{P}) = 0$ for all Null players $i \in N$. This axiom excludes solidarity with unproductive players.

Note that for $\mathcal{P} = \{N\}$, the axioms **GN**, **CE** and **CS** become **N**, **E** and **S** for TU-games (without further structure). These axioms together with **A** characterize the Shapley value (Shapley, 1953).

**Splitting (SP)** A partition $\mathcal{P}' \subseteq 2^N$ is called finer than $\mathcal{P} \subseteq 2^N$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all players $i \in N$. A CS-value $\varphi$ satisfies **SP**, if for $\mathcal{P}'$ being finer than $\mathcal{P}$ we have for all $i \in N$, $j \in \mathcal{P}'(i)$: $\varphi_i(N, v, \mathcal{P}) - \varphi_i(N, v, \mathcal{P}') = \varphi_j(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}')$.

One could argue that gains or losses of splitting a coalition structure should be distributed equally on players staying together in the new coalition structure. Splitting a given coalition structure should affect all players that remain together in the new coalition structure by the same way.

Using the previous axioms, Casajus (2009a) constructs a CS-value that is sensitive to outside options:

**Definition 2.1** ($\chi$-value). *The $\chi$-value is for any player $i \in N$ given by*

$$\chi_i(N, v, \mathcal{P}) := Sh_i(N, v) + \frac{v(\mathcal{P}(i)) - Sh_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|} \tag{2.2}$$

The $\chi$-value can be seen as "the Shapley value made component efficient".

**Theorem 2.2** (Casajus (2009)). *The $\chi$-value is the unique CS-value that satisfies **CE**, **CS**, **A**, **GN** and **SP**.*

# 3 The probabilistic Setting and the probabilistic $\chi$-value

To model a probabilistic coalition situation, consider a probability distribution on all coalition structures $\mathcal{P}$ of $N$. Let $\mathbb{P}(N)$ denote the set of all possible partitions of $N$ and

$$\Delta(\mathbb{P}(N)) := \left\{ p : \mathbb{P}(N) \longrightarrow [0,1]\,,\; \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) = 1 \right\}$$

the set of all probability distributions on $\mathbb{P}(N)$. An element $p \in \Delta(\mathbb{P}(N))$ can be interpreted as $p(\mathcal{P})$ giving the probability that the coalition structure $\mathcal{P}$ occurs.

Define a probabilistic CS-game (*pCS-game or also probabilistic coalition situation*) as a triple $(N, v, p)$, where $(N, v)$ is a (TU) game and $p \in \Delta(\mathbb{P}(N))$. A probabilistic CS-value (pCS-value) $\varphi$ assigns payoff vectors for all pCS-games.

Further, denote for any $p \in \Delta(\mathbb{P}(N))$ by $\mathbb{P}(p) := \{\mathcal{P} \in \mathbb{P}(N) | p(\mathcal{P}) > 0\}$ the set of all coalition

structures whose probability to occur under probability distribution $p$ is not zero (i.e. the carrier of $p$).
A probability distribution $p \in \Delta(\mathbb{P}(N))$ is called *degenerated* if there exists a $\mathcal{P}^* \in \mathbb{P}(N)$ such that $p(\mathcal{P}^*) = 1$ (i.e. $p(\mathcal{P}) = 0\ \forall \mathcal{P} \neq \mathcal{P}^*$). As a notation write $p_{\mathcal{P}^*}$ for the degenerated probability distribution corresponding to the partition $\mathcal{P}^*$.
Identifying $p_{\mathcal{P}}$ with the corresponding partition $\mathcal{P}$, define for every pCS-value $\varphi$ the corresponding CS-value via $\varphi^{det}(N, v, \mathcal{P}) := \varphi(N, v, p_{\mathcal{P}})$. One could interpret this as follows: A degenerated probability distribution gives that a certain coalition structure $\mathcal{P}$ occurs with probability 1 (i.e. $\mathcal{P}$ is the sure event), therefore one is not any longer operating in a probabilistic coalition situation, since there is a surely occuring coalition structure. The other way around, a pCS-value can be defined via a CS-value as a probability-weighted sum. In this situation it is clear that the pCS-value coincides with the CS-value for degenerated probability distributions.

Following this idea I define the probabilistic $\chi$-value as follows:

**Definition 3.1.** *For every probabilistic coalition situation $(N, v, p)$ the probabilistic $\chi$-value of $(N, v, p)$ is defined by:*

$$\chi^p(N, v, p) := \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \chi(N, v, \mathcal{P}) \tag{3.3}$$

Note that in fact $\chi^p(N, v, p_{\mathcal{P}}) = \chi(N, v, \mathcal{P})$.

An extension of CS-axioms into pCS-axioms on degenerated probability distributions is quite straight forward by just replacing $\varphi(N, v, \mathcal{P})$ by $\varphi(N, v, p_{\mathcal{P}})$. Following this I define the axioms **dCE**, **dCS**, **dGN**, **dSP** (where the $d$ refers to an axiom on degenerated probability distributions). Note that having a pCS-value satisfying **dCE**, **dCS**, **dGN**, **dSP** implies that the corresponding CS-value satisfies the deterministic axioms.

Now I introduce a pCS-axiom which fills the gap between degenerated probability distributions and general ones.

**Axiom 3.2** (Linearity on Probability Distributions (pL))**.** *A pCS-value $\varphi$ satisfies **pL** if we have for all probability distributions $p, q \in \Delta(\mathbb{P}(N))$ and all $\alpha \in [0, 1]$:*

$$\varphi(N, v, \alpha p + (1 - \alpha)q) = \alpha\varphi(N, v, p) + (1 - \alpha)\varphi(N, v, q)$$

This axiom states that mixing probabilities should lead to the same mix for the corresponding payoffs. Note that convex combinations of probability distributions are again probability distributions. Mixing probability distributions in a non-convex way would not make sense in this setting.

**Remark 3.3.** If a pCS-value $\varphi$ satisfies **pL**, we have for all probability distributions $p_1, ..., p_m \in$

$\Delta(\mathbb{P}(N))$ and all $\alpha_1, ...\alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$:

$$\varphi\left(N, v, \sum_{i=1}^m \alpha_i p_i\right) = \sum_{i=1}^m \alpha_i \varphi(N, v, p_i)$$

The proof is straight forward by induction over $m$.

**Remark 3.4.** Any probability distribution $p \in \Delta(\mathbb{P}(N))$ can be written as a convex combination of degenerated probability distributions: Since the player set $N$ is finite, $\mathbb{P}(N)$ is also finite. One can write every $p \in \Delta(\mathbb{P}(N))$ as

$$p(\mathcal{P}) = p(\mathcal{P})p_{\mathcal{P}}(\mathcal{P}) = \sum_{\mathcal{P}' \in \mathbb{P}(N)} p(\mathcal{P}')p_{\mathcal{P}'}(\mathcal{P}), \qquad \mathcal{P} \in \mathbb{P}(N)$$
$$\text{with} \sum_{\mathcal{P}' \in \mathbb{P}(N)} p(\mathcal{P}') = 1$$

**Theorem 3.5** (Characterization via deg. prob. distr.)**.** *The probabilistic $\chi$-value is the unique pCS-value that satisfies **dCE**, **dCS**, **A**, **dGN**, **dSP** and **pL**.*

*Proof.* The probabilstic $\chi$-value satisfies **pL**:

$$\begin{aligned}\chi^p(N, v, \alpha p + (1-\alpha)q) &= \sum_{\mathcal{P} \in \mathbb{P}(N)} (\alpha p + (1-\alpha)q)(\mathcal{P})\chi(N, v, \mathcal{P}) \\ &= \alpha\chi^p(N, v, p) + (1-\alpha)\chi^p(N, v, q)\end{aligned}$$

Since $\chi^p(N, v, p_{\mathcal{P}}) = \chi(N, v, \mathcal{P})$, it also satisfies the other axioms by the deterministic value satisfying the deterministic pendants.

Let now $\varphi$ satisfy **dCE**, **dCS**, **A**, **dGN**, **dSP** and **pL**. Define the corresponding (deterministic) CS-value via $\varphi^{det}(N, v, \mathcal{P}) := \varphi(N, v, p_{\mathcal{P}})$.
By this construction, $\varphi^{det}$ satisfies **CE**, **CS**, **A**, **GN** and **SP**. Therefore, by Theorem 2.2, we have $\varphi^{det} = \chi$. Now we get:

$$\begin{aligned}\varphi(N, v, p) &\overset{\text{Remark 3.4}}{=} \varphi\left(N, v, \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})p_{\mathcal{P}}\right) \overset{pL}{=} \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\varphi(N, v, p_{\mathcal{P}}) \\ &= \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\varphi^{det}(N, v, \mathcal{P}) = \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\chi(N, v, \mathcal{P}) = \chi^p(N, v, p)\end{aligned}$$

□

**Remark 3.6.** If a pCS-value is defined as a probability-weighted sum of a deterministic value, it is characterized by **pL** and the degenerated versions of the characterizing axioms of the deterministic value. The proof is following the same structure as in the last proof, just for general values.

Linearity on Probability Distributions is a very strong axiom. We will try to relax **pL** and find another characterization of the probabilistic $\chi$-value. For a characterization without the Linearity-axiom, probabilistic axioms defined for general probability distributions are needed.

# 4 A probabilistic Characterization of the $\chi$-value

## 4.1 Probabilistic Axioms

For every coalition function $v$ define the corresponding probabilistic coalition function $v^p$ as follows:

$$v^p(K) := \sum_{\mathcal{P}\in\mathbb{P}(N)} p(\mathcal{P}) \sum_{S\in\mathcal{P}|_K} v(S)$$

where

$$\mathcal{P}|_K := \{C \cap K | C \in \mathcal{P}\} \setminus \{\emptyset\}$$

Denote by $\mathcal{P}_p$ the finest common coarsening of all $\mathcal{P} \in \mathbb{P}(p)$. A component of $\mathcal{P}_p$, let us say $\mathcal{P}_p(i)$, can be interpreted as the set of all players $j \in N$ that are "connected via probability" with player i. Note that $\mathcal{P}_p = \{N\}$ for many $p \in \Delta(\mathbb{P}(N))$. Also note that $j \in \mathcal{P}_p(i)$ does not necessarily imply the existence of a partition $\mathcal{P} \in \mathbb{P}(p)$ with $j \in \mathcal{P}(i)$.
Now I can define a probabilistic version of **CE**:

**Axiom 4.1** (Probabilistic Component Efficieny (pCE))**.** *A pCS-value $\varphi$ satisfies* ***pCE*** *if $\varphi_C(N, v, p) = v^p(C)$ for all $C \in \mathcal{P}_p$.*

Referring to the productive unit interpretation of **CE**, one can see $\mathcal{P}_p$ as a probabilistic unit that should produce the probabilistic output.

Further, denote by $\mathcal{P}^p$ the coarsest common refinement of all $\mathcal{P} \in \mathbb{P}(p)$. One can interpret a component $\mathcal{P}^p(i) \in \mathcal{P}^p$ as the set of all players that are in the same component as player $i$ for sure, i.e. for any player $j \in \mathcal{P}^p(i)$ we have that $j \in \mathcal{P}(i)\ \forall\ \mathcal{P} \in \mathbb{P}(p)$.
With this I define a probabilistic version of **CS**:

**Axiom 4.2** (Probabilistic Symmetry within Components (pCS))**.** *A pCS-value $\varphi$ satisfies* ***pCS*** *if we have $\varphi_i(N, v, p) = \varphi_j(N, v, p)$ for all players $i, j$ being symmetric in $(N, v)$, $j \in \mathcal{P}^p(i)$.*

Players with the same productivity that are in the same component for sure should not be treated differently in the distribution of the payoff.

With the following axiom I give a probabilistic version of **SP**:

**Axiom 4.3** (Probabilistic Splitting (pSP))**.** *A pCS-value $\varphi$ satisfies* ***pSP*** *if for $\mathcal{P}', \mathcal{P} \in \mathbb{P}(N)$, $\mathcal{P}'$ finer than $\mathcal{P}$, and for $p, q \in \Delta(\mathbb{P}(N))$ with $q(\mathcal{P}'') = p(\mathcal{P}'')\ \forall\ \mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}', \mathcal{P}\}$ we have $\varphi_i(N, v, p) - \varphi_i(N, v, q) = \varphi_j(N, v, p) - \varphi_j(N, v, q)$ for all $i \in N$, $j \in \mathcal{P}'(i)$.*

At first view, this axiom might look very constructed. But looking more carefully at the requirements of the probability distributions $p$ and $q$ in the axiom one sees that one just shifts probability between a partition $\mathcal{P}$ and some finer partition $\mathcal{P}'$. For the axiom **SP** I argued that gains or losses of splitting a coalition structure should affect all players that remain together by the same way. Now one could argue that if one shifts probability between a certain coalition structure and a split coalition structure, players that would stay together in the split coalition structure should be affected by the same way.

Note that **pSP** implies **dSP** by taking $p$ and $q$ as the degenerated probability distributions corresponding to $\mathcal{P}$ and $\mathcal{P}'$ respectively. One has $p_{\mathcal{P}}(\mathcal{P}'') = 0 = p_{\mathcal{P}'}(\mathcal{P}'')\ \forall\ \mathcal{P}'' \in \mathbb{F}(N) \setminus \{\mathcal{P}, \mathcal{P}'\}$, therefore I can apply **pSP**.
If axioms have requirements on probability distributions like in **pSP**, consider the following algorithm in order to work on the whole set $\Delta(\mathbb{P}(N))$.

**Algorithm 4.4** (Shift-Algorithm)**.** Every $p \in \Delta(\mathbb{P}(N))$ can be constructed from $p_{\{N\}}$ by stepwise shifting probability between a partition $\mathcal{P}$ and a finer partition $\mathcal{P}'$:
Take $p \in \Delta(\mathbb{P}(N))$ arbitrary. Since $N$ is finite, $\mathbb{P}(N)$ is also finite, therefore I can order the partitions: $\mathbb{P}(p) = \{\mathcal{P}_1, ..., \mathcal{P}_n\}$. If $\{N\} \in \mathbb{P}(p)$, set $\{N\} = \mathcal{P}_n$.
Consider the following algorithm: I start with $q_0 = p_{\{N\}}$. For all $i \in \{1, ..., n-1\}$ I set:

$$q_i(\mathcal{P}_i) = p(\mathcal{P}_i),\ q_i(\{N\}) = 1 - \sum_{j=1}^{i} p(\mathcal{P}_j) \text{ and}$$
$$q_i(\mathcal{P}'') = q_{i-1}(\mathcal{P}'')\ \forall\ \mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}_i, \{N\}\}$$

Note that the last part implies $q_i(\mathcal{P}_j) = 0\ \forall\ j > i,\ \mathcal{P}_j \neq \{N\}$ and also $q_i(\mathcal{P}_j) = p(\mathcal{P}_j)\ \forall\ j < i$.
Case 1: $\{N\} \notin \mathbb{P}(p)$. I follow the recursive construction from above:

$$\begin{aligned} &q_n(\mathcal{P}_n) = p(\mathcal{P}_n),\ q_n(\{N\}) = 1 - \sum_{j=1}^{n} p(\mathcal{P}_j) = 0 \text{ and} \\ &\forall\ \mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}_n, \{N\}\} : q_n(\mathcal{P}'') = q_{n-1}(\mathcal{P}'') = p(\mathcal{P}'') \\ \Rightarrow &q_n = p \end{aligned}$$

Case 2: $\{N\} \in \mathbb{P}(p) (\Leftrightarrow \{N\} = \mathcal{P}_n)$. Consider $q_{n-1}$:

$$\begin{aligned} &q_{n-1}(\mathcal{P}_{n-1}) = p(\mathcal{P}_{n-1}),\ q_{n-1}(\{N\}) = 1 - \sum_{j=1}^{n-1} p(\mathcal{P}_j) = 1 - (1 - p(\{N\})) = p(\{N\}), \\ &\forall\ \mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}_{n-1}, \mathcal{P}_n\} : q_{n-1}(\mathcal{P}'') = q_{n-2}(\mathcal{P}'') = p(\mathcal{P}''), \\ \Rightarrow &q_{n-1} = p \end{aligned}$$

Every step from some $q_i$ to $q_{i+1}$ is a shift of probability from $\{N\}$ to $\mathcal{P}_{i+1}$. Since every $\mathcal{P} \in \mathbb{P}(p)$ is finer than $\{N\}$, the algorithm describes a construction of an arbitrary $p \in \Delta(\mathbb{P}(N))$ by stepwise shifting probability between a partition and a finer one.

**Lemma 4.5.** *The probabilistc $\chi$-value satisfies **pCE**, **pCS** and **pSP**.*

*Proof.* **pCE**: Take $C \in \mathcal{P}_p$. Note that for every $\mathcal{P} \in \mathbb{P}(p)$ one has

$$\mathcal{P}|_C = \{C' \in \mathcal{P} : \bigcup_{C' \in \mathcal{P}} = C\} \tag{$*$}$$

Using that the deterministic $\chi$-value satisfies **CE** I get:

$$\begin{aligned}
\chi_C(N,v,p) &= \sum_{i \in C} \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \chi_i(N,v,\mathcal{P}) = \sum_{\mathcal{P} \in \mathbb{P}(p)} p(\mathcal{P}) \sum_{i \in C} \chi_i(N,v,\mathcal{P}) \\
&\overset{(*)}{=} \sum_{\mathcal{P} \in \mathbb{P}(p)} p(\mathcal{P}) \sum_{S \in \mathcal{P}|_C} \sum_{i \in S} \chi_i(N,v,\mathcal{P}) \\
&\overset{\text{CE}}{=} \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \sum_{S \in \mathcal{P}|_C} v(S) = v^p(C)
\end{aligned}$$

**pCS**: Let $i,j$ be symmetric in $(N,v)$, $j \in \mathcal{P}^p(i)$, i.e. $j \in \mathcal{P}(i)\ \forall\ \mathcal{P} \in \mathbb{P}(p)$. Using that the deterministic $\chi$-value satisfies **CS** I get:

$$\chi_i(N,v,p) = \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \chi_i(N,v,\mathcal{P}) \overset{\text{CS}}{=} \sum_{\mathcal{P} \in \mathbb{P}(p)} p(\mathcal{P}) \chi_j(N,v,\mathcal{P}) = \chi_j(N,v,p)$$

**pSP**: Let $\mathcal{P}'$ be finer than $\mathcal{P}$ and $p,q \in \Delta(\mathbb{P}(N))$ such that $p(\mathcal{P}'') = q(\mathcal{P}'')\ \forall\ \mathcal{P}'' \in \mathbb{P} \setminus \{\mathcal{P}', \mathcal{P}\}$. First note that

$$\begin{aligned}
p(\mathcal{P}') - q(\mathcal{P}') &= \left(1 - \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N) \setminus \{\mathcal{P}'\}} p(\bar{\mathcal{P}})\right) - \left(1 - \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N) \setminus \{\mathcal{P}'\}} q(\bar{\mathcal{P}})\right) \\
&= - \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N) \setminus \{\mathcal{P}'\}} [p(\bar{\mathcal{P}}) - q(\bar{\mathcal{P}})] = -[p(\mathcal{P}) - q(\mathcal{P})]
\end{aligned} \tag{$**$}$$

Let $i \in N$. Using that deterministic $\chi$-value satisfies **SP** one has for all $j \in \mathcal{P}'(i)$:

$$\begin{aligned}
\chi_i(N,v,p) - \chi_i(N,v,q) &= \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} p(\bar{\mathcal{P}}) \chi_i(N,v,\bar{\mathcal{P}}) - \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} q(\bar{\mathcal{P}}) \chi_i(N,v,\bar{\mathcal{P}}) \\
&= \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} [p(\bar{\mathcal{P}}) - q(\bar{\mathcal{P}})] \chi_i(N,v,\bar{\mathcal{P}}) \\
&= [p(\mathcal{P}) - q(\mathcal{P})] \chi_i(N,v,\mathcal{P}) + [p(\mathcal{P}') - q(\mathcal{P}')] \chi_i(N,v,\mathcal{P}') \\
&\overset{(**)}{=} [p(\mathcal{P}) - q(\mathcal{P})][\chi_i(N,v,\mathcal{P}) - \chi_i(N,v,\mathcal{P}')] \\
&\overset{\text{SP}}{=} [p(\mathcal{P}) - q(\mathcal{P})][\chi_j(N,v,\mathcal{P}) - \chi_j(N,v,\mathcal{P}')] \\
&\overset{\text{backwards}}{=} \chi_j(N,v,p) - \chi_j(N,v,q)
\end{aligned}$$

□

I now try to characterize the probabilistic $\chi$-value by my new axioms.

**Conjecture 4.6.** *The probabilistic $\chi$-value is the unique pCS-value that satsfies **pCE**, **pCS**, **A**, **dGN** and **pSP**.*

However this conjecture is wrong. Another plausible way to define a probabilistic value is using the probabilistic coalition function instead of the probability-weighted sum. Using the probabilistic component one gets:

$$\varphi_i(N, v, p) := Sh_i(N, v) + \frac{v^p(\mathcal{P}_p(i)) - Sh_{\mathcal{P}_p(i)}(N, v)}{|\mathcal{P}_p(i)|}$$

It is easy to see that $\varphi$ satisfies **A**, **dGN**, **pCE** and **pCS**. To see **pSP**, let $\mathcal{P}, \mathcal{P}' \in \mathbb{P}(N)$ and $p, q \in \Delta(\mathbb{P}(N))$, $p \neq q$, meet the requirements of **pSP**. Take $i \in N$, $j \in \mathcal{P}'(i)$. Since $\mathcal{P}'$ is finer than $\mathcal{P}$, we have $j \in \mathcal{P}(i)$. Consider 3 cases:

1. $p(\mathcal{P}') > 0 \Rightarrow j \in \mathcal{P}_p(i)$
2. $p(\mathcal{P}') = 0 \wedge p(\mathcal{P}) > 0 \Rightarrow j \in \mathcal{P}_p(i)$
3. $p(\mathcal{P}') = 0 = p(\mathcal{P}) \Rightarrow p = q$ contradiction!

Analogously one gets $j \in \mathcal{P}_q(i)$. Hence, $\mathcal{P}_p(i) = \mathcal{P}_p(j), \mathcal{P}_q(i) = \mathcal{P}_q(j)$, which gives **pSP**.
Consider the following probabilistic coalition structure:
$N = \{1, 2, 3\}, v = u_{\{2,3\}}, \mathcal{P} = \{\{1, 2\}, \{3\}\}, p(\{N\}) = p(\mathcal{P}) = \frac{1}{2}$.
Consider the Null player 1 (every player $i \in N \setminus T$ is a Null player in the unanimity game $u_T$). Using that the Shapley value satisfies **N** one can calculate:

$$\chi_1(N, u_{\{2,3\}}, p) = -\frac{1}{8} \text{ and } \varphi_1(N, u_{\{2,3\}}, p) = -\frac{1}{6} \neq -\frac{1}{8}$$

Hence, $\varphi \neq \chi$ and both satisfy the axioms, i.e. $\chi$ is not the unique value satisfying these axioms.

One could construct $\varphi$ also by following the idea of the proof of the deterministic case (Theorem 2.2). At some point one has to assume $\mathcal{P}^p(i) = \mathcal{P}_p(i)$ (which is generally not the case) in order to be able to use **pCE**. In a lot of cases $P_p(i)$ contains all players in $N$. Hence, in a sense, **pCE** is too weak!

## 4.2 Influence Axiom

I go back to the use of **dCE** instead of **pCE**. To fill the thereby created gap I introduce an axiom which controles the influence of a probability shift.

**Axiom 4.7** (Probabilistic Influence on Components (pIC))**.** *A pCS-value $\varphi$ satisfies **pIC** if for $\mathcal{P}', \mathcal{P} \in \mathbb{P}(N)$, $\mathcal{P}'$ finer than $\mathcal{P}$, and for $p, q \in \Delta(\mathbb{P}(N))$ with $q(\mathcal{P}'') = p(\mathcal{P}'')$ $\forall$ $\mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}', \mathcal{P}\}$ we have*

$$\varphi_P(N, v, p) - \varphi_P(N, v, q) = [p(\mathcal{P}') - q(\mathcal{P}')][\varphi_P(N, v, p_{\mathcal{P}'}) - \varphi_P(N, v, p_{\mathcal{P}})]$$

*for all $P \in \mathcal{P}'$.*

This axiom controls how a shift of probability between a given coalition structure and a finer one influences the payoff of a component of the finer coalition structure.

**Theorem 4.8** (Characterization of the probabilistic $\chi$-value)**.** *The probabilistic $\chi$-value is the unique pCS-value that satisfies **pIC**, **dCE**, **dCS**, **A**, **dGN** and **pSP**.*

*Proof.* The probabilistic $\chi$-value satisfies **pIC**: Let $\mathcal{P}', \mathcal{P} \in \mathbb{P}(N)$ and $p, q \in \Delta(\mathbb{P}(N))$ satisfy the requirements of **pIC**. Using equality $(**)$ from the proof of Lemma 4.5, part 3 one has for all $P \in \mathcal{P}'$:

$$\begin{aligned}
\chi_P^p(N,v,p) - \chi_P^p(N,v,q) &= \sum_{i \in P} \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} [p(\bar{\mathcal{P}}) - q(\bar{\mathcal{P}})] \chi_i(N,v,\bar{\mathcal{P}}) \\
&\overset{(**)}{=} \sum_{i \in P} [p(\mathcal{P}') - q(\mathcal{P}')][\chi_i(N,v,\mathcal{P}') - \chi_i(N,v,\mathcal{P})] \\
&= [p(\mathcal{P}') - q(\mathcal{P}')][\chi_P(N,v,\mathcal{P}') - \chi_P(N,v,\mathcal{P})] \\
&= [p(\mathcal{P}') - q(\mathcal{P}')][\chi_P^p(N,v,p_{\mathcal{P}'}) - \chi_P^p(N,v,p_{\mathcal{P}})]
\end{aligned}$$

It has already been shown that the probabilistic $\chi$-value satisfies the other axioms.
Let now $\varphi$ satisfy **pIC**, **dCE**, **dCS**, **A**, **dGN** and **pSP**. First note that $\varphi(N,v,p_{\{N\}}) = Sh(N,v)$[2]. I proceed proving uniqueness by induction over $k = |\mathbb{P}(p) \setminus \{N\}|$.[3]
For the case $k = 0$ we have $|\mathbb{P}(p) \setminus \{N\}| = 0$ which implies $p = p_{\{N\}}$ and therefore implies $\varphi(N,v,p) = Sh(N,v)$.
Now, suppose $\varphi(N,v,p) = \chi(N,v,p)$ for all $p \in \Delta(\mathbb{P}(N))$ such that $|\mathbb{P}(p) \setminus \{N\}| = k$ $(H)$.
For the inductive step $k \mapsto k+1$ take $p$ with $|\mathbb{P}(p) \setminus \{N\}| = k+1$ arbitrary. Construct $p$ via the Algorithm 4.4. There are two cases:

1. $\mathbb{P}(p) = \{\mathcal{P}_1, ..., \mathcal{P}_{k+1}\}, \{N\} \notin \mathbb{P}(p) \Rightarrow q_{k+1} = p$
2. $\mathbb{P}(p) = \{\mathcal{P}_1, ..., \mathcal{P}_{k+1}, \{N\}\} \Rightarrow q_{(k+2)-1} = q_{k+1} = p$

Hence, both cases give the same $(q_i)_{i=0,...,k+1}$. Note that $|\mathbb{P}(q_k) \setminus \{N\}| = k$.
Every $\mathcal{P} \in \mathbb{P}(N)$ is finer than $\{N\}$. By construction I have $q_k(\mathcal{P}) = q_{k+1}(\mathcal{P})\ \forall\ \mathcal{P} \in \mathbb{P}(N) \setminus \{\mathcal{P}_{k+1}, \{N\}\}$. Therefore, by **pSP**, I have for $i \in N$, $j \in \mathcal{P}_{k+1}(i)$:

$$\varphi_i(N,v,q_{k+1}) - \varphi_i(N,v,q_k) = \varphi_j(N,v,q_{k+1}) - \varphi_j(N,v,q_k)$$

Summing up over $j \in \mathcal{P}_{k+1}(i)$:

$$|\mathcal{P}_{k+1}(i)|[\varphi_i(N,v,q_{k+1}) - \varphi_i(N,v,q_k)] = \varphi_{\mathcal{P}_{k+1}(i)}(N,v,q_{k+1}) - \varphi_{\mathcal{P}_{k+1}(i)}(N,v,q_k) \tag{4.4}$$

By the construction of $(q_i)_i$ and $(H)$ I get for the left hand side of (4.4):

$$|\mathcal{P}_{k+1}(i)|[\varphi_i(N,v,p) - \chi_i^p(N,v,q_k)]$$

[2] For $p_{\{N\}}$ the axioms **pCE**, **pCS** and **dGN** become **dE**, **dS** and **dN**, identify $\varphi(N,v,p_{\{N\}})$ with the deterministic value $\varphi(N,v,\{N\})$.
[3] $|\mathbb{P}(p) \setminus \{N\}| \geq 0$ because $\mathbb{P}(p) \geq 1$ since $p$ is a probability distribution.

By **pIC** and the use of **dCE** I get for the right hand side of (4.4):

$$\begin{aligned}
&[q_{k+1}(\mathcal{P}_{k+1}) - q_k(\mathcal{P}_{k+1})][\varphi_{\mathcal{P}_{k+1}(i)}(N, v, p_{\mathcal{P}_{k+1}}) - \varphi_{\mathcal{P}_{k+1}(i)}(N, v, p_{\{N\}})] \\
\overset{dCE}{=}&[q_{k+1}(\mathcal{P}_{k+1}) - q_k(\mathcal{P}_{k+1})][v(\mathcal{P}_{k+1}(i)) - \varphi_{\mathcal{P}_{k+1}(i)}(N, v, p_{\{N\}})] \\
=&[p(\mathcal{P}_{k+1}) - 0][v(\mathcal{P}_{k+1}(i)) - Sh_{\mathcal{P}_{k+1}(i)}(N, v)]
\end{aligned}$$

Together one gets:

$$|\mathcal{P}_{k+1}(i)|[\varphi_i(N, v, p) - \chi_i^p(N, v, q_k)] = p(\mathcal{P}_{k+1})[v(\mathcal{P}_{k+1}(i)) - Sh_{\mathcal{P}_{k+1}(i)}(N, v)]$$

Hence, for all $i \in N$, we get

$$\varphi_i(N, v, p) = \chi_i^p(N, v, q_k) + p(\mathcal{P}_{k+1}) \frac{[v(\mathcal{P}_{k+1}(i)) - Sh_{\mathcal{P}_{k+1}(i)}(N, v)]}{|\mathcal{P}_{k+1}(i)|}$$

which uniquely determines $\varphi$. □

## 4.3 Linearity Equivalent

My aim was not only to find another characterization of the probabilistic $\chi$-value, I also wanted to relax the Linearity axiom **pL**. The axioms in Theorem 4.8 are already similar to the Linearity-Characterization axioms of Theorem 3.5, I just use **pSP** instead of **dSP** and **pIC** instead of **pL**. Is there a connection between **pIC** and **pL**?

For that I first introduce an axiom which obviously strengthens the Probabilistic Influence on Components axiom **pIC**.

**Axiom 4.9** (Probabilistic Influence on Players (pIP))**.** *A pCS-value $\varphi$ satisfies **pIP** if for $p, q \in \Delta(\mathbb{P}(N))$ with $q(\mathcal{P}'') = p(\mathcal{P}'')$ $\forall$ $\mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}', \mathcal{P}\}$ for some $\mathcal{P}, \mathcal{P}' \in \mathbb{P}(N)$ we have*

$$\varphi_i(N, v, p) - \varphi_i(N, v, q) = [p(\mathcal{P}') - q(\mathcal{P}')][\varphi_i(N, v, p_{\mathcal{P}'}) - \varphi_i(N, v, p_{\mathcal{P}})]$$

*for all $i \in N$.*

This axiom controls the influence of a probability shift between any two partitions $\mathcal{P}$ and $\mathcal{P}'$ on a player's payoff. Note that **pIP** obviously implies **pIC**. Also note that this axiom is not only applicable for $\mathcal{P}'$ being finer than $\mathcal{P}$, but for *any* $\mathcal{P}', \mathcal{P} \in \mathbb{P}(N)$.

Rewriting the right hand side of **pIP** gives

$$\varphi_i(N, v, p) - \varphi_i(N, v, q) = \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} [p(\bar{\mathcal{P}}) - q(\bar{\mathcal{P}})]\varphi_i(N, v, p_{\bar{\mathcal{P}}})$$

i.e. **pIP** states that the payoff difference due to a probability-shift between two coalition structures equals the payoff difference of the probability-weighted sums of the deterministic value.

**Theorem 4.10** (Linearity Equivalent)**.** *The axiom* ***pIP*** *is equivalent to the axiom* ***pL****.*

*Proof.* "$\Leftarrow$" Let $\varphi$ satisfy **pL**. Let $\mathcal{P}, \mathcal{P}'$ and $p, q$ satisfy the requirements of **pIP**. Using $[p(\mathcal{P}') - q(\mathcal{P}')] = -[p(\mathcal{P}) - q(\mathcal{P})]$[4] I have:

$$\begin{aligned}\varphi_i(N,v,p) - \varphi_i(N,v,q) &\overset{pL}{=} \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} p(\bar{P})\varphi_i(N,v,p_{\bar{\mathcal{P}}}) - \sum_{\bar{\mathcal{P}} \in \mathbb{P}(N)} q(\bar{P})\varphi_i(N,v,p_{\bar{\mathcal{P}}}) \\ &= [p(\mathcal{P}') - q(\mathcal{P}')][\varphi_i(N,v,p_{\mathcal{P}'}) - \varphi_i(N,v,p_{\mathcal{P}})]\end{aligned}$$

"$\Rightarrow$" Let $\varphi$ satisfy **pIP**. Let $\alpha < 1$ (the case $\alpha = 1$ is trivial).

The proof will contain three steps

1. **pL** for $p, q$ degenerated probability distributions:

$$\varphi(N,v,\alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'}) = \alpha\varphi(N,v,p_{\mathcal{P}}) + (1-\alpha)\varphi(N,v,p_{\mathcal{P}'})$$

2. **pL** for general $p \in \Delta(\mathbb{P}(N))$, $q$ degenerated probability distribution:

$$\varphi(N,v,\alpha p + (1-\alpha)p_{\mathcal{P}^*}) = \alpha\varphi(N,v,p) + (1-\alpha)\varphi(N,v,p_{\mathcal{P}^*})$$

3. **pL** for general $p, q \in \Delta(\mathbb{P}(N))$

For 1. notice that:

$$\begin{aligned}&(\alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'})(\mathcal{P}'') = 0 = p_{\mathcal{P}}(\mathcal{P}'')\ \forall \mathcal{P}'' \in \mathbb{P}(N) \setminus \{\mathcal{P}, \mathcal{P}'\} \\ &(\alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'})(\mathcal{P}) = \alpha < 1 = p_{\mathcal{P}}(\mathcal{P}) \\ &(\alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'})(\mathcal{P}') = 1 - \alpha > 0 = p_{\mathcal{P}}(\mathcal{P}')\end{aligned}$$

Hence, $p_{\mathcal{P}}$ and $q := \alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'}$ satisfy the requirements of **pIP**.

$$\begin{aligned}\varphi(N,v,q) - \varphi(N,v,p_{\mathcal{P}}) &\overset{\text{pIP}}{=} [q(\mathcal{P}') - p_{\mathcal{P}}(\mathcal{P}')][\varphi(N,v,p_{\mathcal{P}'}) - \varphi(N,v,p_{\mathcal{P}})] \\ &= (1-\alpha)[\varphi(N,v,p_{\mathcal{P}'}) - \varphi(N,v,p_{\mathcal{P}})] \\ \Leftrightarrow \varphi(N,v,\alpha p_{\mathcal{P}} + (1-\alpha)p_{\mathcal{P}'}) &= \alpha\varphi(N,v,p_{\mathcal{P}}) + (1-\alpha)\varphi(N,v,p_{\mathcal{P}'})\end{aligned}$$

Using this result I have for all $p \in \Delta(\mathbb{P}(N))$:[5]

$$\varphi(N,v,p) = \varphi\left(N, v, \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})p_{\mathcal{P}}\right) = \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\varphi(N,v,p_{\mathcal{P}}) \qquad (*)$$

For 2. set $\mathbb{P}(p) := \{\mathcal{P}_1, ..., \mathcal{P}_n\}$ with $|\mathbb{P}(p)| > 1$, i.e. $p$ not degenerated. During the proof I will have to distinguish between two cases: $\mathcal{P}^* \notin \mathbb{P}(p)$ and $\mathcal{P}^* := \mathcal{P}_n \in \mathbb{P}(p)$.

[4] cf. proof of Lemma 4.5, part 3 equation $(**)$
[5] cf. Remark 3.3 and Remark 3.4.

Note for $\mathcal{P}^* = \mathcal{P}_n \in \mathbb{P}(p)$

$$(\alpha p + (1-\alpha)p_{\mathcal{P}^*})(\mathcal{P}_i) = \alpha p(\mathcal{P}_i) < p(\mathcal{P}_i) \;\forall\, i = 1, ..., n-1$$
$$(\alpha p + (1-\alpha)p_{\mathcal{P}^*})(\mathcal{P}^*) = \alpha p(\mathcal{P}^*) + 1 - \alpha > \alpha p(\mathcal{P}^*) + (1-\alpha)p(\mathcal{P}^*) = p(\mathcal{P}^*)$$

and for $\mathcal{P}^* \notin \mathbb{P}(p)$

$$(\alpha p + (1-\alpha)p_{\mathcal{P}^*})(\mathcal{P}_i) = \alpha p(\mathcal{P}_i) < p(\mathcal{P}_i) \;\forall\, i = 1, ..., n$$
$$(\alpha p + (1-\alpha)p_{\mathcal{P}^*})(\mathcal{P}^*) = 1 - \alpha > 0 = p(\mathcal{P}^*)$$

For both cases consider the following algorithm:
I start with $q_0 := p$
For all $i = 1, ..., n-1$:

$$q_i(\mathcal{P}') = q_{i-1}(\mathcal{P}') \;\forall\; \mathcal{P}' \in \mathbb{P}(N) \setminus \{\mathcal{P}_i, \mathcal{P}^*\},$$
$$q_i(\mathcal{P}_i) = \alpha p(\mathcal{P}_i),$$
$$q_i(\mathcal{P}^*) = q_{i-1}(\mathcal{P}^*) + (1-\alpha)p(\mathcal{P}_i) = p(\mathcal{P}^*) + (1-\alpha)\sum_{j=1}^{i} p(\mathcal{P}_j) > 0$$

Note that $q_i(\mathcal{P}_j) = p(\mathcal{P}_j) \;\forall\; j > i$, $\mathcal{P}_j \neq \mathcal{P}^*$.

For case $\mathcal{P}^* = \mathcal{P}_n \in \mathbb{P}(p)$ I consider $q_{n-1}$:

$$q_{n-1}(\mathcal{P}') = q_{n-2}(\mathcal{P}') = \alpha p(\mathcal{P}') \;\forall\; \mathcal{P}' \in \mathbb{P}(N) \setminus \{\mathcal{P}_{n-1}, \mathcal{P}^*\},$$
$$q_{n-1}(\mathcal{P}_{n-1}) = \alpha p(\mathcal{P}_{n-1}),$$
$$q_{n-1}(\mathcal{P}^*) = p(\mathcal{P}^*) + (1-\alpha)\underbrace{\sum_{j=1}^{n-1} p(\mathcal{P}_j)}_{=1-p(\mathcal{P}^*)} = \alpha p(\mathcal{P}^*) + 1 - \alpha$$

$$\Rightarrow q_{n-1} = \alpha p + (1-\alpha)p_{\mathcal{P}^*}$$

and $q_n$ for $\mathcal{P}^* \notin \mathbb{P}(p)$:

$$q_n(\mathcal{P}') = q_{n-1}(\mathcal{P}') = \alpha p(\mathcal{P}') \;\forall\; \mathcal{P}' \in \mathbb{P}(N) \setminus \{\mathcal{P}_n, \mathcal{P}^*\},$$
$$q_n(\mathcal{P}_n) = \alpha p(\mathcal{P}_n),$$
$$q_n(\mathcal{P}^*) = \underbrace{p(\mathcal{P}^*)}_{=0} + (1-\alpha)\underbrace{\sum_{j=1}^{n} p(\mathcal{P}_j)}_{=1} = 1 - \alpha$$

$$\Rightarrow q_n = \alpha p + (1-\alpha)p_{\mathcal{P}^*}$$

For every $i = 1, ..., n-1$ I get by **pIP**:

$$\begin{aligned}\varphi(N,v,q_i) - \varphi(N,v,q_{i+1}) &= [q_i(\mathcal{P}_{i+1}) - q_{i+1}(\mathcal{P}_{i+1})][\varphi(N,v,p_{\mathcal{P}_{i+1}}) - \varphi(N,v,p_{\mathcal{P}^*})] \\ &= [p(\mathcal{P}_{i+1}) - \alpha p(\mathcal{P}_{i+1})][\varphi(N,v,p_{\mathcal{P}_{i+1}}) - \varphi(N,v,p_{\mathcal{P}^*})] \\ \Leftrightarrow \varphi(N,v,q_{i+1}) &= \varphi(N,v,q_i) - (1-\alpha)p(\mathcal{P}_{i+1})[\varphi(N,v,p_{\mathcal{P}_{i+1}}) - \varphi(N,v,p_{\mathcal{P}^*})]\end{aligned}$$

Using **pIP** $(n-1)$ times I get for $\mathcal{P}^* = \mathcal{P}_n \in \mathbb{P}(p)$:

$$\begin{aligned}\varphi(N,v,q_{n-1}) &= (1-\alpha)p(\mathcal{P}_{n-2})[\varphi(N,v,p_{\mathcal{P}_{n-2}}) - \varphi(N,v,p_{\mathcal{P}^*})] + \varphi(N,v,q_{n-2}) \\ &\dots \\ &= \varphi(N,v,q_0) - \sum_{i=1}^{n-1}(1-\alpha)p(\mathcal{P}_i)[\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})] \\ &= \varphi(N,v,p) - (1-\alpha)\left[\sum_{i=1}^{n-1} p(\mathcal{P}_i)\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})\sum_{i=1}^{n-1} p(\mathcal{P}_i)\right] \\ &= \varphi(N,v,p) - (1-\alpha)\left[\sum_{i=1}^{n-1} p(\mathcal{P}_i)\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})(1-p(\mathcal{P}^*))\right] \\ &\overset{\mathcal{P}^*=\mathcal{P}_n}{=} \varphi(N,v,p) - (1-\alpha)\left[\sum_{i=1}^{n} p(\mathcal{P}_i)\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})\right] \\ &\overset{(*)}{=} \varphi(N,v,p) - (1-\alpha)[\varphi(N,v,p) - \varphi(N,v,p_{\mathcal{P}^*})] \\ &= \alpha\varphi(N,v,p) + (1-\alpha)\varphi(N,v,p_{\mathcal{P}^*})\end{aligned}$$

Analogously, using **pIP** $n$ times, I get for $\mathcal{P}^* \notin \mathbb{P}(p)$:

$$\begin{aligned}\varphi(N,v,q_n) &= \varphi(N,v,q_0) - \sum_{i=1}^{n}(1-\alpha)p(\mathcal{P}_i)[\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})] \\ &= \varphi(N,v,p) - (1-\alpha)\left[\sum_{i=1}^{n} p(\mathcal{P}_i)\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})\sum_{i=1}^{n} p(\mathcal{P}_i)\right] \\ &= \varphi(N,v,p) - (1-\alpha)\left[\sum_{i=1}^{n} p(\mathcal{P}_i)\varphi(N,v,p_{\mathcal{P}_i}) - \varphi(N,v,p_{\mathcal{P}^*})\right] \\ &\overset{(*)}{=} \varphi(N,v,p) - (1-\alpha)[\varphi(N,v,p) - \varphi(N,v,p_{\mathcal{P}^*})] \\ &= \alpha\varphi(N,v,p) + (1-\alpha)\varphi(N,v,p_{\mathcal{P}^*})\end{aligned}$$

Hence, for both cases, one gets **pL**.

For 3. proceed showing **pL** for general $p, q \in \Delta(\mathbb{P}(N))$ by induction on $n = |\mathbb{P}(q)|$. For notational reasons write $\varphi(p) := \varphi(N,v,p)$ for $(N,v)$ fixed.
For the case $n = 1$ one has $q = p_{\mathcal{P}^*}$ for some $\mathcal{P}^* \in \mathbb{P}(N)$. ✓ by 2.

Now, suppose $\varphi(\alpha p + (1-\alpha)q) = \alpha\varphi(p) + (1-\alpha)\varphi(q)$ for all $q$ s.th. $|\mathbb{P}(q)| = n$ (H).

For the inductive step $n \mapsto n+1$ set $\mathbb{P}(q) = \{\mathcal{P}_1, ..., \mathcal{P}_{n+1}\}$. Use the Shift Algorithm (cf. Lemma 4.4), but take $\mathcal{P}^* := \mathcal{P}_{n+1}$ instead of $\{N\}$. Therefore $q_0 = p_{\mathcal{P}^*}$ and $q_n = q$ since $\mathcal{P}^* \in \mathbb{P}(p)$.

First note that $q_{n-1}(\mathcal{P}_n) = 0$ and $q_{n-1}(\mathcal{P}) \neq 0 \; \forall \; \mathcal{P} \neq \mathcal{P}_n$, hence $|\mathbb{P}(q_{n-1})| = n$. Then note that $(\alpha p + (1-\alpha)q_n)(\mathcal{P}) = (\alpha p + (1-\alpha)q_{n-1})(\mathcal{P}) \; \forall \; \mathcal{P} \in \mathbb{P}(N) \setminus \{\mathcal{P}_n, \mathcal{P}^*\}$, hence I can use **pIP**.

$$\begin{aligned}
\varphi(\alpha p + (1-\alpha)q_n) &\overset{\text{pIP}}{=} [\alpha p(\mathcal{P}_n) + (1-\alpha)q_n(\mathcal{P}_n) - \alpha p(\mathcal{P}_n) - (1-\alpha)q_{n-1}(\mathcal{P}_n)] \\
&\quad \cdot [\varphi(p_{\mathcal{P}_n}) - \varphi(p_{\mathcal{P}^*})] + \varphi(\alpha p + (1-\alpha)q_{n-1}) \\
&= (1-\alpha)q_n(\mathcal{P}_n)[\varphi(p_{\mathcal{P}_n}) - \varphi(p_{\mathcal{P}^*})] + \varphi(\alpha p + (1-\alpha)q_{n-1}) \\
&\overset{(H)}{=} (1-\alpha)q_n(\mathcal{P}_n)[\varphi(p_{\mathcal{P}_n}) - \varphi(p_{\mathcal{P}^*})] + \alpha\varphi(p) + (1-\alpha)\varphi(q_{n-1}) \\
&= \alpha\varphi(p) + (1-\alpha)[\varphi(q_{n-1}) + [q_n(\mathcal{P}_n) - 0][\varphi(p_{\mathcal{P}_n}) - \varphi(p_{\mathcal{P}^*})]] \\
&= \alpha\varphi(p) + (1-\alpha)[\underbrace{\varphi(q_{n-1}) + [q_n(\mathcal{P}_n) - q_{n-1}(\mathcal{P}_n)][\varphi(p_{\mathcal{P}_n}) - \varphi(p_{\mathcal{P}^*})]}_{=\varphi(q_n) \text{ by pIP}}] \\
&= \alpha\varphi(p) + (1-\alpha)\varphi(q_n)
\end{aligned}$$

$\Rightarrow$ **pIP** $\Leftrightarrow$ **pL**. □

Using the equivalence of **pIP** and **pL** in the degenerated characterization of the probabilistic $\chi$-value (Theorem 4.8), one trivially gets the following corollary:

**Corollary 4.11.** The probabilistic $\chi$-value is the unique pCS-value that satisfies **dCE**, **dCS**, **A**, **dGN**, **dSP** and **pIP**.

Since **pIC** relaxes **pIP**, it is also a relaxation of **pL**. Therefore, Theorem 4.8 is the characterization of the probabilistic $\chi$-value that relaxes the Linearity-Characterization (Theorem 3.5) I was looking for.

Note here that the gap emerging due to the relaxation of the Linearity axiom is filled by taking the probabilistic version of the Splitting axiom.

# 5 Component Efficiency and the AD-value

It has been shown that **pCE** is not sufficient for a characterization of the probabilistic $\chi$-value. Gómez et al. (2008) define a probabilistic version of the Myerson-value (Myerson, 1977) and show that it can be characterized by probabilistic pendants of the characterizing deterministic axioms. This characterization includes a version of **pCE** for network structures. The graph-$\chi$-value (Casajus 2009b), an outside-option-sensitive value for network structures, can be seen as the Myerson value "made outside-option sensitive".
It turns out that the approach of characterizing a probabilistic version of the graph-$\chi$-value via

**pCE** seems to fail as well as for the $\chi$-value[6]. Therefore, I wonder whether **pCE** is sufficient for a characterization of the "outside-option-insensitive pendant" of the $\chi$-value, the component restricted Shapley value: Aumann and Drèze (1974) define the component restricted Shapley value (I denote it by Aumann-Drèze value $AD$) for every CS-game $(N, v, \mathcal{P})$ as follows

$$AD_i(N, v, \mathcal{P}) := Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)})$$

Myerson (1977) defines the following property:

**Axiom 5.1** (Balanced Contributions (BC)). *A CS-value $\varphi$ satisfies **BC**, if*

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i(N \setminus \{j\}, v|_{N\setminus\{j\}}, \mathcal{P}|_{N\setminus\{j\}}) = \varphi_j(N, v, \mathcal{P}) - \varphi_j(N \setminus \{i\}, v|_{N\setminus\{i\}}, \mathcal{P}|_{N\setminus\{i\}})$$

*for all $i, j \in N$.*

The exit of a player $j$ should hurt/benefit another player $i$ by the same amount as the exit of $i$ hurts/benefits $j$.

Myerson (1977) also shows that the Shapley value is characterized by **E** and **BC**. Slikker and van den Nouweland (2001) prove that the Aumann-Drèze value is the unique CS-value that satisfies **CE** and **CBC**, where **CBC** is the restriction of **BC** on components (i.e. $j \in \mathcal{P}(i)$). If a value satisfies **BC**, it particularly satisfies **CBC**. Since the Aumann-Drèze-value not only satisfies **CBC** but also **BC**, we can characterize $AD$ by **CE** and **BC**:

**Theorem 5.2.** *The AD-value is the unique CS-value that satisfies **CE** and **BC**.*

*Proof.* $AD$ satisfies **CE** by $Sh$ satisfying **E**. For **BC** let $i, j \in N$ and one has

$$\begin{aligned}
&AD_i(N \setminus \{j\}, v|_{N\setminus\{j\}}, \mathcal{P}|_{N\setminus\{j\}}) - AD_j(N \setminus \{i\}, v|_{N\setminus\{i\}}, \mathcal{P}|_{N\setminus\{i\}}) \\
&= Sh_i(\mathcal{P}|_{N\setminus\{j\}}(i), v|_{N\setminus\{j\}}|_{\mathcal{P}|_{N\setminus\{j\}}(i)}) - Sh_j(\mathcal{P}|_{N\setminus\{i\}}(j), v|_{N\setminus\{i\}}|_{\mathcal{P}|_{N\setminus\{i\}}(j)}) \\
&= Sh_i(\mathcal{P}(i) \setminus \{j\}, v|_{\mathcal{P}(i)\setminus\{j\}}) - Sh_j(\mathcal{P}(j) \setminus \{i\}, v|_{\mathcal{P}(j)\setminus\{i\}}) \\
&\overset{(*)}{=} Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}) - Sh_j(\mathcal{P}(j), v|_{\mathcal{P}(j)}) \\
&= AD_i(N, v, \mathcal{P}) - AD_j(N, v, \mathcal{P})
\end{aligned}$$

$(*)$: If $j \in \mathcal{P}(i)$ use that $Sh$ satisfies **BC**. If $j \notin \mathcal{P}(i)$ one also has $i \notin \mathcal{P}(j)$ and therefore $\mathcal{P}(i) \setminus \{j\} = \mathcal{P}(i)$ and $\mathcal{P}(j) \setminus \{i\} = \mathcal{P}(j)$.

Let now $\varphi$ satisfy **CE** and **BC**. Using that **BC** implies **CBC** leads to the fact that $\varphi$ also satisfies **CBC**. Hence, $\varphi$ is uniquely determined (using the characterization of Slikker and van den Nouweland (2001)). □

I now define the probabilistic Aumann-Drèze value and a probabilistic version of **BC**. Again I use the probability-weighted sum:

[6] Further information can be requested from the author

**Definition 5.3.** *For every probabilistic coalition situation $(N, v, p)$ the probabilistic Aumann-Drèze value of $(N, v, p)$ is defined by:*

$$AD^p(N, v, p) := \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) AD(N, v, \mathcal{P})$$

For the probabilistic version of **BC** the probability distribution corresponding to the restricted playerset is needed. For every $p \in \Delta(\mathbb{P}(N))$ and every $\mathcal{P}' \in \mathbb{P}(N \setminus \{i\})$, $i \in N$, I define

$$p^{-i}(\mathcal{P}') := \sum_{\substack{\mathcal{P} \in \mathbb{P}(N) \\ \mathcal{P}' = \mathcal{P}|_{N \setminus \{i\}}}} p(\mathcal{P})$$

With this preliminary considerations I define:

**Axiom 5.4** (probabilistic Balanced Contributions (pBC)). *A CS-value $\varphi$ satisfies **pBC**, if*

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i(N \setminus \{j\}, v|_{N \setminus \{j\}}, p^{-j}) = \varphi_j(N, v, \mathcal{P}) - \varphi_j(N \setminus \{i\}, v|_{N \setminus \{i\}}, p^{-i})$$

*for all $i, j \in N$.*

**Theorem 5.5** (Characterization of the probabilistic *AD*-value). *The probabilistic AD-value is the unique pCS-value that satisfies **pCE** and **pBC**.*

*Proof.* I show **pCE** analogously to the proof for the probabilistic $\chi$-value (Lemma 4.5): Take $C \in \mathcal{P}_p$. Note that for every $\mathcal{P} \in \mathbb{P}(p)$ one has

$$\mathcal{P}|_C = \{C' \in \mathcal{P} : \bigcup_{C' \in \mathcal{P}} = C\} \tag{$*$}$$

Using that the deterministic *AD*-value satisfies **CE** I get:

$$\begin{aligned} AD^p_C(N, v, p) &= \sum_{i \in C} \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) AD_i(N, v, \mathcal{P}) = \sum_{\mathcal{P} \in \mathbb{P}(p)} p(\mathcal{P}) \sum_{i \in C} AD_i(N, v, \mathcal{P}) \\ &\overset{(*)}{=} \sum_{\mathcal{P} \in \mathbb{P}(p)} p(\mathcal{P}) \sum_{S \in \mathcal{P}|_C} \sum_{i \in S} AD_i(N, v, \mathcal{P}) \overset{\text{CE}}{=} \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \sum_{S \in \mathcal{P}|_C} v(S) \\ &= v^p(C) \end{aligned}$$

To see **pBC** take $i, j \in N$. Using that the deterministic *AD*-avlue satisfies **BC** I get:

$$\begin{aligned} & AD^p_i(N, v, p) - AD^p_i(N \setminus \{j\}, v|_{N \setminus \{j\}}, p^{-j}) \\ &= \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) AD_i(N, v, \mathcal{P}) - \sum_{\mathcal{P}' \in \mathbb{P}(N \setminus \{j\})} p^{-j}(\mathcal{P}') AD_i(N \setminus \{j\}, v|_{N \setminus \{j\}}, \mathcal{P}') \\ &= \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) AD_i(N, v, \mathcal{P}) - \sum_{\mathcal{P}' \in \mathbb{P}(N \setminus \{j\})} \sum_{\substack{\mathcal{P} \in \mathbb{P}(N) \\ \mathcal{P}' = \mathcal{P}|_{N \setminus \{j\}}}} p(\mathcal{P}) AD_i(N \setminus \{j\}, v|_{N \setminus \{j\}}, \mathcal{P}') \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathcal{P}\in\mathbb{P}(N)} p(\mathcal{P})AD_i(N,v,\mathcal{P}) - \sum_{\substack{\mathcal{P}\in\mathbb{P}(N)\\ \mathcal{P}|_{N\setminus\{j\}}=\mathcal{P}'}} \sum_{\mathcal{P}'\in\mathbb{P}(N\setminus\{j\})} p(\mathcal{P})AD_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, \mathcal{P}') \\
&= \sum_{\mathcal{P}\in\mathbb{P}(N)} p(\mathcal{P})AD_i(N,v,\mathcal{P}) - \sum_{\mathcal{P}\in\mathbb{P}(N)} p(\mathcal{P})AD_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, \mathcal{P}|_{N\setminus\{j\}}) \\
&= \sum_{\mathcal{P}\in\mathbb{P}(N)} \left[AD_i(N,v,\mathcal{P}) - AD_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, \mathcal{P}|_{N\setminus\{j\}})\right] \\
&\overset{BC}{=} \sum_{\mathcal{P}\in\mathbb{P}(N)} \left[AD_j(N,v,\mathcal{P}) - AD_j(N\setminus\{i\}, v|_{N\setminus\{i\}}, \mathcal{P}|_{N\setminus\{i\}})\right] \\
&\overset{\text{backwards}}{=} AD_j^p(N,v,p) - AD_j^p(N\setminus\{i\}, v|_{N\setminus\{i\}}, p^{-i})
\end{aligned}
$$

Let now $\varphi$ satisfy the axioms. I prove uniqueness by induction over $n = |N|$.
For the case $n = 1$ one has $N = \{i\}$, i.e. $\Delta(\mathbb{P}(\{N\})) = \{p_{\{N\}}\}$. For $p = p_{\{N\}}$ the axioms become **dE** and **dBC**[7] respectively. Since $Sh$ is characterized by **E** and **BC** one has $\varphi_i(N,v,p_{\{N\}}) = Sh_i(N,v) = Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}) = AD_i^p(N,v,p_{\{N\}})$.

Now, suppose $\varphi(N,v,p) = AD^p(N,v,p)$ for all $(N,v,p)$ with $|N| = n$ $(H)$.

For the inductive step $n \mapsto n+1$ take $i \in N$. If $|\mathcal{P}_p(i)| = 1$ one has $\{i\} \in \mathcal{P}_p$, i.e. $\{i\} \in \mathcal{P}$ for all $\mathcal{P} \in \mathbb{P}(p)$ and therefore

$$
\begin{aligned}
\varphi_i(N,v,p) &\overset{CE}{=} v^p(\{i\}) = v(\{i\}) \overset{E}{=} Sh_i(\{i\}, v|_{\{i\}}) \\
&= \sum_{\mathcal{P}\in\mathbb{P}} p(\mathcal{P}) Sh_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}) = AD_i^p(N,v,p)
\end{aligned}
$$

Let now $|\mathcal{P}_p(i)| \leq 2$. By **pBC** I have for all $j \in N$, particularly for all $j \in \mathcal{P}_p(i)$:

$$
\varphi_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, p^{-j}) - \varphi_j(N\setminus\{i\}, v|_{N\setminus\{i\}}, p^{-i}) = \varphi_i(N,v,p) - \varphi_j(N,v,p) \tag{5.5}
$$

Summing up over $j \in \mathcal{P}_p(i)\setminus\{i\}$ I have for the right hand side of (5.5):

$$
\begin{aligned}
&\sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} [\varphi_i(N,v,p) - \varphi_j(N,v,p)] \\
&= (|\mathcal{P}_p(i)| - 1)\varphi_i(N,v,p) - \sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} \varphi_j(N,v,p) \\
&= |\mathcal{P}_p(i)|\varphi_i(N,v,p) - \varphi_{\mathcal{P}_p(i)}(N,v,p) \\
&\overset{pCE}{=} |\mathcal{P}_p(i)|\varphi_i(N,v,p) - v^p(\mathcal{P}_p(i))
\end{aligned}
$$

[7] the degenerated version of **BC**

and for the left hand side of (5.5):

$$\begin{aligned}
&\sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} \left[\varphi_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, p^{-j}) - \varphi_j(N\setminus\{i\}, v|_{N\setminus\{i\}}, p^{-i})\right] \\
&\overset{(H)}{=} \sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} \left[AD^p_i(N\setminus\{j\}, v|_{N\setminus\{j\}}, p^{-j}) - AD^p_j(N\setminus\{i\}, v|_{N\setminus\{i\}}, p^{-i})\right] \\
&\overset{pBC}{=} \sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} \left[AD^p_i(N, v, p) - AD^p_j(N, v, p)\right] \\
&= (|\mathcal{P}_p(i)| - 1)AD^p_i(N, v, p) - \sum_{j\in\mathcal{P}_p(i)\setminus\{i\}} AD^p_j(N, v, p) \\
&= |\mathcal{P}_p(i)|AD^p_i(N, v, p) - AD^p_{\mathcal{P}_p(i)}(N, v, p) \\
&\overset{pCE}{=} |\mathcal{P}_p(i)|AD^p_i(N, v, p) - v^p(\mathcal{P}_p(i))
\end{aligned}$$

Combining these results I get $\varphi_i(N, v, p) = AD^p_i(N, v, p)$ for all $i \in N$. □

Hence, the probabilistic $AD$-value can be characterized directly via probabilistic pendants of the characterizing determinstic axioms. Unlike for the probabilistic $\chi$-value, **pCE** is sufficient. Why is possible to translate the deterministic characterization of the $AD$-value directly into a probabilistic one while it is possible for the outside-option-sensitive pendant? A possible explanation might be the component decomposability of the Aumann-Drèze value:

**Axiom 5.6** (Component Decomposability (CD)). *A CS-value $\varphi$ satisfies **CD**, if for all $i \in N$*

$$\varphi_i(N, v, \mathcal{P}) = \varphi_i(\mathcal{P}(i), v|_{\mathcal{P}(i)}, \{\mathcal{P}(i)\})$$

If a value is component decomposable, it seems to be reasonable that the difference of the probabilistic components (cf. chapter 4.1, page 14) should not matter and one should not run into problems with probabilistic Component Efficiency. I expect problems with probabilistic Component Efficiency for a direct probabilistic characterization of values that are not component decomposable while I think that component decomposable values should have a direct probabilistic characterization. Note here that both the Aumann-Drèze value and the Myerson value are component decomposable while their outside-option-sensitive pendants are not.

Should we think about an outside-option-sensitive value that is component decomposable in order to get a "nice" axiomatization? Component Decomposability states that the player's outside world does not affect payoffs within a component. Neither the potential coalitions between players in the component and outside the component, nor the coalition structure. Hence, it stands in contradiction to outside-option-sensitivity! In other words, even if we take another outside-option-sensitive value, it can not be component decomposable and hence it will most probably not have a direct probabilistic characterization.

# 6 Conclusion and Critcism

I defined the setting of probabilistic coalition situations, which is more general than the already known setting of (deterministic) coalition situations and useful for forecasting aims. I found a probabilistic extension of the outside-option-sensitive $\chi$-value and gave an axiomatic characterization via Linearity on Probability Distributions and degenerated versions of the characterizing axioms of the deterministic value. This linearity axiom is very strong, therefore I wanted to relax this axiom and find a further characterization. Even though it was not possible to find a characterization via probabilistic pendants of the deterministic axioms directly, I succeeded in finding the desired relaxation and characterization via introducing the influence axioms.
In the end I showed that the problem of the insufficiency of Component Efficiency for characterizing the $\chi$-value is not any longer present for the outside-option-insensitive pendant. Further I claimed that outside-option-sensitive values generally lack a direct probabilistic characterization.

The analyzed concept is a probability-weighted sum of deterministic pendants and therefore the expected ex ante payoff; one could argue that a deterministic characterization is sufficient. But it has been shown in this paper that it is not possible to translate the deterministic characterization directly into a probabilistic one.
The network-approach for modeling social or economic situations captures more information about the structure of the society or economy than the coalition-structure-approach. Hence, it seems to be a more adequate approach. Casajus (2009b) introduces an outside-option-sensitive value for networks. However, for this value it does not seem to be possible to find a characterization in which one relaxes the Linearity axiom[8]. It would be interesting to find a probabilistic outside-option-sensitive value for networks which can be characterized via probabilistic axioms.

[8] More informations can be requested from the author

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