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A Nonparametric Analysis

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Jan Heufer[1]

Quasiconcave Preferences and Choices on a Probability Simplex – A Nonparametric Analysis

Abstract

A nonparametric approach is presented to test whether decisions on a probability simplex could be induced by quasiconcave preferences. Necessary and sufficient conditions are presented. If the answer is affirmative, the methods developed here allow to reconstruct bounds on indifference curves. Furthermore we can construct quasiconcave utility functions in analogy to the utility function constructed in the proof of Afriat's Theorem. The approach is of interest for decisions under risk, stochastic choice, and ex-ante fairness considerations. The method is particularly suitable for data collected in a laboratory experiment.

JEL Classification: C14, C91, D11, D12, D81

Keywords: Afriat's theorem; deterministic preferences; decisions under risk; experimental economcics; nonparametric methods; revealed preference; stochastic choice.

March 2010

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1 Technische Universität Dortmund. – I am grateful to my advisor Wolfgang Leininger for his support and comments. Thanks to Anthony la Grange and Burkhard Hehenkamp for helpful comments. Views expressed represent exclusively the author's own opinions. – All correspondence to Jan Heufer, Department of Economics and Social Science, Technische Universität Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany. e-mail: jan.heufer@uni-dortmund.de.

1 INTRODUCTION

The expected utility (EU) hypothesis implies that an individual's indifference curves in a probability space are straight parallel lines. Empirical evidence, however, shows that this is generally not the case. Allais and Hagen (1979), Kahneman and Tversky (1979), Morrison (1967), and Sopher and Narramore (2000) are just some examples of compelling evidence that indifference curves systematically deviate from parallelness and straightness.

A related yet different topic are considerations for ex ante fairness. Suppose an individual can choose a point on a subset of a probability simplex that represents probabilities of different consumers for winning a prize. An individual (the dictator in an experimental setting) can give up some of his own probability of winning the prize in exchange for a fairer ex ante allocation. Such preferences for fairness have been considered by Karni and Safra (2002*a*) and Karni and Safra (2002*b*) and experimentally investigated by Karni et al. (2008). The theoretical analysis implies that individuals with preferences for fairness have quasiconcave preferences in the probabilities; the experimental analysis indicates that this is often the case.

This paper is concerned with a nonparametric approach to the analysis of decisions on a probability simplex. The questions are (i) when individuals make decisions on subsets of a probability simplex, what testable conditions can be found to refute the hypothesis that individuals have quasiconcave preference on the probability simplex, and (ii) if the hypothesis is not refuted, how can we reconstruct bounds on the indifference curves? A key reference here is Machina (1985) who considers implications on choice behavior when preferences are quasiconcave. The paper is also in the spirit of Varian's (1982, 1983) nonparametric approach to demand behavior and the experimental approach of Choi et al. (2007*b*; see also Choi et al. 2007*a* and Fisman et al. 2007).

The rest of the paper is organized as follows. Section 2 reviews two of the most relevant models for the framework considered here, specifically stochastic choice functions generated by deterministic preferences over lotteries and considerations for ex ante fairness. Section 3 introduces the notation and shows how to determine which part of the probability simplex is revealed worse to an observation. This is used to redefine budgets by means of an implicit function, a concept which will be crucial for the proof in the next section. Section 4 uses the results of the previous section to show the close analogy to the revealed preference approach for usual commodity spaces. Three axioms are presented which closely resemble the Weak (Samuelson 1938), Strong (Houthakker 1950), and Generalized (Afriat 1967, Varian 1982) Axiom of Revealed Preference. The section gives constructive proofs in anlogy to Afriat's Theorem to show that consistency with our Generalized (Strong) Axiom is equivalent to the existence of a (strictly) quasiconcave utility function which rationalizes the observations. Section 5 discusses further possibilities of analyzing observations and of reconstructing bounds on indifference curves through unobserved points. Section 6 illustrates the approach with examples and shows how to approaximate the power of the test for quasiconcavity by Monte Carlo experiments in the spirit of Bronars (1987). Section 7 concludes.

2.1 *Stochastic Choice Generated by Deterministic Preferences over Lotteries*

Stochastic choice has been studied by many researchers in the psychological and also in the economic literature. Early examples include Block and Marschak (1960) and Becker et al. (1963); Machina (1985) provides a list of references. More recently, stochastic choices on linear budgets have been analyzed by Bandyopadhyay et al. (1999), Bandyopadhyay et al. (2002), Bandyopadhyay et al. (2004), and Heufer (2008). The basic idea is that individuals have unstable or random preferences, or some important factors that influence choice are unobservable to the researcher and the choice behavior therefore appears to have stochastic components. As Machina (1985) states,

> [t]he motivation for such an approach is clear: if when confronted with a choice over two objects the individual chooses each alternative a positive proportion of the time, it seems natural to suppose that this is because he or she 'prefers' each one to the other those same proportions of the time.

Machina, in the same paper, then goes on to provide "an alternative model of stochastic choice at the individual level". He assumes that individuals do not have stochastic preferences over pure outcomes but rather deterministic preferences over lotteries. If an individual chooses option A with probability p over option B, then he does not prefer A over B p proportion of time, but rather the individual actually prefers a lottery that yields A with probability p over any pure outcome.

Machina's interpretation of "stochastic choice" as deterministic preferences over lotteries has been experimentally tested against the aforementioned hypothesis by Sopher and Narramore (2000). They find support for Machina's idea; they report that

> [i]n general, subjects prefer mixtures of lotteries over extremes [...] Moreover, they are consistent over time, in the sense that the distribution of choices (for a given linear choice set) does not change very often [...] We interpret these results as supporting the deterministic preference version of stochastic choice over the random utility interpretation.

Quasiconcave preference, i.e. preferences for randomization, have also been considered in Crawford (1990), Chew et al. (1991), Camerer (1992), Camerer and Ho (1994), and Starmer (2000), but the most detailed analysis of its implications can still be found in Machina (1985).

2.2 *Individual Preferences for Ex Ante Fairness*

Karni and Safra (2002*a*) (see also Karni and Safra (2002*b*)) provide an axiomatic model of the behavior of an individual with both self interest and preferences for fairness. The individual chooses a random allocation procedure; preferences for fairness imply convex indifference curves in a probability simplex, i.e. quasiconcave preferences. Imagine an experiment with three subject, one being a "dictator" who has to divide an indivisible good by assigning winning probabilities to each subject. A dictator with strong preferences for ex ante fairness might prefer $\{p_1, p_2, p_3\} = \{⅓, ⅓, ⅓\}$, whereas a selfish dictator might prefer $\{p_1, p_2, p_3\} = \{1, 0, 0\}$. Karni et al. (2008) investigate choice behavior in such an experiment by offering subjects "budgets", i.e. line segments in the probability simplex.

3 REVEALED PREFERENCES AND BUDGET SPECIFICATION

3.1 *The Basics*

This paper is concerned with decisions on hyperplanes as subsets of a set of lotteries when preferences are quasiconcave. First, we would like to find refutable conditions on observed choices which are hypothesized as generated by quasiconcave preferences. As will be seen later, we can not only find necessary and sufficient conditions for the existence of a quasiconcave utility function which rationalizes the data, we can even construct such a utility function using a generalization of Afriat's (1967) theorem due to Forges and Minelli (2009). Second, we would like to reconstruct boundaries on the indifference curves in the simplex which are implied by the observed choices. Arguably one of the best ways to observe meaningful data in this framework is in a laboratory setting. We therefore suggest to interpret the analysis here as an outline for experimental research.

Following Machina (1985), individuals are assumed to possess a utility function $V : D(A) \to \mathbb{R}$, where

$$D(A) = \left\{ (p_1, \ldots, p_n) : p_i \in [0,1], \sum_{i=1}^{n} p_i = 1 \right\} \tag{1}$$

is a set of lotteries over a set $A = \{a_1, \ldots, a_n\}$ of distinct pure outcomes; the interior of $D(A)$ is

$$\text{int}D(A) = \left\{ (p_1, \ldots, p_n) : p_i \in (0,1), \sum_{i=1}^{n} p_i = 1 \right\}. \tag{2}$$

The individual's choice probabilities over any subset of A correspond to that lottery over the subset of A which maximizes $V(\cdot)$. We assume that generally the set of available alternatives is of the form

$$D_A(B) = \left\{ \sum_{i=1}^{n-1} \lambda_i b^i : \lambda_i \in [0,1], \sum_{i=1}^{n-1} \lambda_i = 1 \right\}, \tag{3}$$

where the elements of the set $B = \{b^1, \ldots, b^{n-1}\}$ are elements of $D(A)$, i.e. $B \subset A$. We will also refer to a set of available alternatives as a "budget". Note that a budget is the convex hull of B. For example, if $A = \{a_1, a_2, a_3\}$, $D(A)$ can be represented as a 2-simplex in which any set of available alternatives is a line segment, as in Figure 1. If $A = \{a_1, \ldots, a_4\}$, a budget can be thought of as a hyperplane segment inside a tetrahedron. As will be seen later, budgets can also be generalized to arbitrary sets in $D(A)$, although it is doubtful whether that would be of any advantage (or even feasible) in an experimental setting.

Given a set $D_A(B)$ of available alternatives, an individual will choose a lottery $x \in D_A(B)$ which maximizes his utility $V(\cdot)$ on $D_A(B)$. The choice correspondence on a

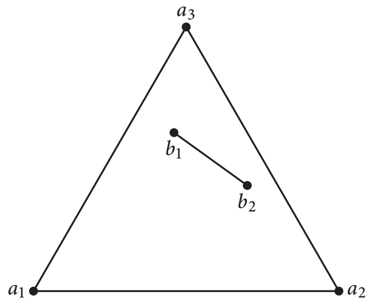

Figure 1: A budget in the simplex.

budget, a sujective mapping $C_A : D(A) \to D(A)$, is defined as that vector of probabilities over the elements of B which generates the most preferred distribution over A, i.e.

$$C_A(B) = \{\arg\max_{p \in D_A(B)} V(p)\}. \tag{4}$$

This is equivalent to stating that the consumer chooses a lottery $x \in D_A(B)$ such that his indifference curve through that point is just tangent to the set $D_A(B)$; see Figure 2 for an example.

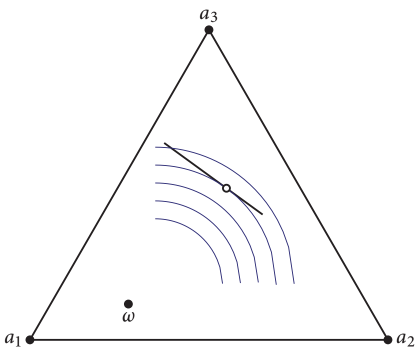

Figure 2: An optimal choice on the budget.

A preference $\succsim \subset D(A) \times D(A)$ is a complete *preorder*[1], i.e. a binary relation which is complete, reflexive and transitive. That means that $\succsim$ is a set of ordered pairs; it is complete if either $(x, y) \in \succsim$ or $(y, x) \in \succsim$ or both; it is reflexive if $(x, x) \in \succsim$; it is transitive if $(x, y) \in \succsim$ and $(y, z) \in \succsim$ imply $(x, z) \in \succsim$; for $(x, y) \in \succsim$ we also write $x \succsim y$. The symmetric part of $\succsim$ is denoted by $\sim$ and its asymmetric part is denoted by $\succ$, i.e. $(x, y) \in \sim$ if $(x, y) \in \succsim$ and $(y, x) \in \succsim$, and $(x, y) \in \succ$ if $(x, y) \in \succsim$ and $(x, y) \notin \succsim$. A preference is *quasiconcave*

[1] A preorder is also called a quasiorder.

if for all $x, y \in D(A)$ $x \sim y$ implies $\lambda x + (1-\lambda) y \succsim y$ for $\lambda \in (0,1)$, or alternatively, if for all $y \in D(A)$ the set $\{x \in D(A) : x \succsim y\}$ is convex. It is *strictly quasiconcave* if $\{x \in \text{int} D(A) : x \succsim y\}$ is strictly convex.

We say that the function $V(\cdot)$ *represents* the preference $\succsim$ if for all $x, y \in D(A)$, $x \succsim y$ implies $V(x) \geq V(y)$ and $x \succ y$ implies $V(x) > V(y)$.

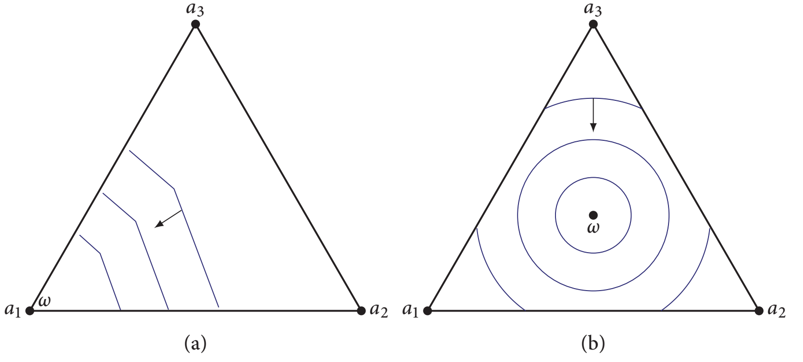

Figure 3: Preferred points and indifference curves for quasiconcave preferences; ω is the most preferred point in the simplex. Arrows denote increasing preference. Left: Preferences are quasiconcave. Right: Preferences are strictly quasiconcave.

Quasiconcavity implies that an individual's indifference curves are convex; see Figure 3 for two examples. We assume that there is a unique $\succsim$-maximal element in $D(A)$, denoted by ω, i.e. in a simplex representing $D(A)$ there is a single most preferred lottery. The preferences we consider are therefore *single-peaked* in $D(A)$, and any utility function which represents a preference is *satiated* at ω and nonsatiated at any $x \in D(A)$ such that $x \neq \omega$.[2] If the elements of A are monetary outcomes, ω will be the degenerated lottery that assigns probability 1 to the maximal element in A (remember that we have assumed that the outcomes are distinct). If outcomes are different modes of transportation, then ω might be in the interior of $D(A)$. If a point in the simplex represents an allocation of winning probabilities of a lottery for different individuals, an individual with strong preferences for ex ante fairness who gets to choose the point might prefer the element with equal probabilities, i.e. the centroid of the simplex. For most of the paper we assume that we know ω, as subjects in an experiment can be easily and incentive-compatibly asked to reveal ω directly. We later discuss ways to analyze decisions when ω is not known.

3.2 *The Revealed Worse Set*

Given $D_A(B)$ and $C_A(B)$, what elements of A are revealed worse than any $x \in C_A(B)$ under the hypothesis of quasiconcave preferences? Let $H_A(B)$ denote the $\mathbb{R}^{n-1}$ hyperplane which contains B.

[2] A utility function V is nonsatiated at $x \in D(A)$ if there exists an $\varepsilon > 0$ such that $d(x,y) > \varepsilon$ and $V(x) \geq V(y)$ for some $y \in D(A)$, where d is the Euclidean distance function; a utility function is satiated at ω if there does not exist an $\varepsilon > 0$ such that $d(\omega, y) > \varepsilon$ and $V(y) \geq V(\omega)$ for any $y \in D(A)$.

Proposition 1. *Suppose $C_A(B) \subseteq \text{int}D_A(B)$. The hyperplane $H_A(B)$ separates $D(A)$ into two disjoint sets. If preferences are (weakly) quasiconcave, then the set $dRW_A(x) \subseteq D(A)$ that is* directly revealed worse *than any $x \in C_A(B)$ is the union of $H_A(B)$ and the set that does not contain ω. The set $sdRW_A(x) \subset D(A)$ that is* strictly directly revealed worse *than $x \in C_A(B)$ is $dRW_A(x) \setminus H_A(B)$.*

Proof Consider Figure 4.(a), where $x \in C_A(B)$. Suppose $y \succ x$. Then by quasiconcavity, $\lambda x + (1-\lambda) y \succ x$ for all $\lambda \in (0,1)$. Suppose $\lambda x + (1-\lambda) y = z \in D_A(B)$, then $z \succ x$, and by $x \in C_A(B)$ we have $x \succsim z$, a contradiction. Obviously, y can be chosen to be any lottery on $H_A(B)$ and λ can be chosen small enough such that $z \in D_A(B)$, so for every $y \in H_A(B)$ we must have that $x \succsim y$. This obviously holds for more than three outcomes as well.

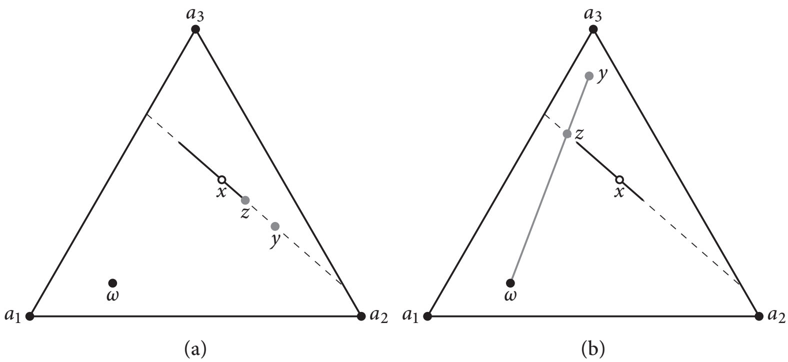

Figure 4: The hyperplane which contains the budget is the boundary of the directly revealed worse set.

Consider Figure 4.(b), where again $x \in C_A(B)$. Choose a lottery y in the set that does not contain ω, i.e. in the postulated set $sdRW_A(x)$. We have $\omega \succ y$. Then by quasiconcavity, $z = \lambda \omega + (1-\lambda) y \succ y$. Suppose $y \succsim x$, then $z \succ x$, which contradicts $x \in C_A(B)$. ∎

Proposition 2. *Suppose $C_A(B) \subseteq \text{int}D_A(B)$. If preferences are strictly quasiconcave, then the set $dRW_A(x) \subseteq D(A)$ that is* directly revealed worse *than any $x \in C_A(B)$ is the union of $H_A(B)$ and the set that does not contain ω. The set $sdRW_A(x) \subset D(A)$ that is* strictly directly revealed worse *than $x \in C_A(B)$ is $dRW_A(x) \setminus x$.*

Proof Consider again Figure 4.(a), where again $x \in C_A(B)$. Suppose $y \succsim x$. By strict quasiconcavity this implies that $z \succ x$, a contradiction. The rest follows from Proposition 1. ∎

Corollary 1 (of Proposition 2). *If preferences are strictly quasiconcave, the set $C_A(B)$ is a singleton and C_A is one-to-one and onto.*

Proof Suppose $\{x, y\} \subseteq C_A(B)$. By Proposition 2 we have $x \succ y$ and $y \succ x$, a contradiction. ■

The directly revealed worse set is illustrated in Figure 5.

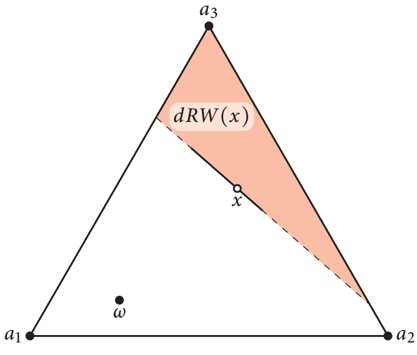

Figure 5: The directly revealed worse set.

The (indirectly) revealed worse set can be constructed in the following way: Let $S_A = \{x^i, B^i\}_{i\in\{1,\ldots,m\}}$ be a set of m observations, where $x^i \in C_A(B^i)$ is the observed choice on $D_A(B^i)$. Suppose $x^j \in dRW_A(x^i)$, so x^j is directly revealed worse than x^i. Then the set $dRW_A(x^j)$ is indirectly revealed worse than x^i. Suppose $x^k \notin dRW_A(x^i)$ but $x^k \in dRW_A(x^j)$. Then the set $dRW_A(x_k)$ is also indirectly revealed worse than x^i. That is, the set $RW_A(x^\ell)$ that is *revealed worse* than x^ℓ is the union of all $dRW_A(x^i)$ for which either $x^i \in dRW_A(x^\ell)$ or for some chain of observations with indices $i, j, k, \ldots, c$, we have $x^i \in dRW_A(x^j)$, $x^j \in dRW_A(x^k)$, …, $x^c \in dRW_A(x^\ell)$, and similarly for the *strictly revealed worse* set. See Figure 6 for an example. We skip the formal proof because it is a straightforward application of transitivity in conjunction with Proposition 1 or 2.

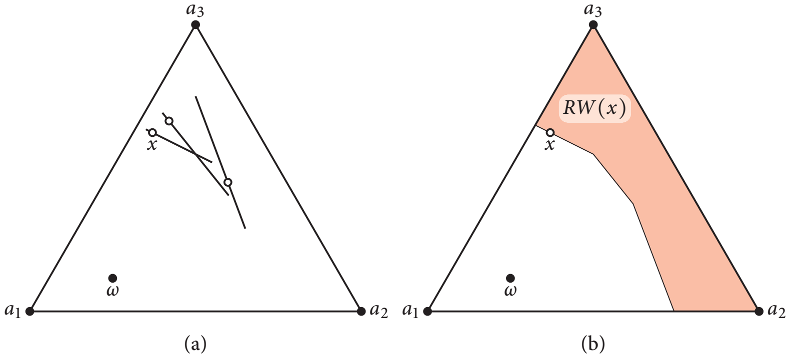

Figure 6: Construction of a revealed worse set with indirect relations.

3.3 *Decisions on the Boundary of a Budget and the Optimal Design of an Experiment*

Suppose that an observed decision $\{x\} = C_A(B)$ is not in $\text{int}D_A(B)$ but on the boundary. In that case the above result that all lotteries on the hyperplane which contains $D_A(B)$ are revealed worse than x is not valid because x might not have been the optimal lottery on that hyperplane. A general analysis would be rather involved in case of more than three possible outcomes; but as will be seen, the problem can easily be avoided in an experimental setting.

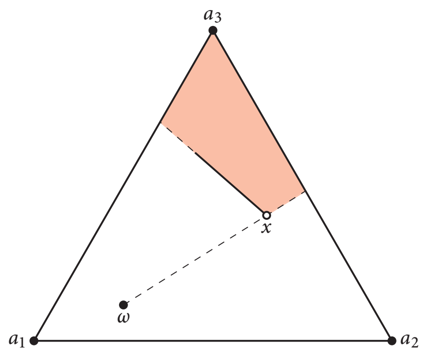

Figure 7: The revealed worse set for a choice on the boundary of the budget.

Consider Figure 7, which depicts the revealed worse set in case x is on the boundary of a budget. The extension of the budget to the upper left follows the arguments from above. Note that any lottery outside the indicated revealed worse set can be contained in a convex set that does also contain ω, but does not intersect the budget or its extension to the upper left. That is, we can draw a convex indifference set with x on the "worse" side, so any lottery outside the indicated revealed worse set could indeed be preferred to x. We cannot do this for a lottery in the revealed worse set; see Figure 8.(a) and also Figure 8.(b) for an extreme example.

As was seen, decisions on the boundary reveal "less information" and are not as easy to interpret as decisions in the interior of a budget. This suggest that a researcher designing an experiment should refrain from constructing budgets from lotteries in the interior of $D(A)$. Instead it appears to be the most promising to construct budgets from lotteries on the boundary of $D(A)$, as depicted in Figure 9. For an example with four possible outcomes, see Figure 10.

3.4 *Budgets Described by Implicit Functions in the Marschak-Machina Triangle*

If budgets are chosen such that for all $b^i \in B \subset D(A)$ we have $b^i \notin \text{int}D(A)$, i.e. budgets are constructed from lotteries on the boundary of $D(A)$ as recommended in Section 3.3, then our definition of budgets admits the possibility of describing a budget as

$$\tilde{B} = \left\{x \in \mathbb{R}_+^{n-1} : \tilde{g}(x) = 0\right\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \leq 1\right\} \quad (5)$$

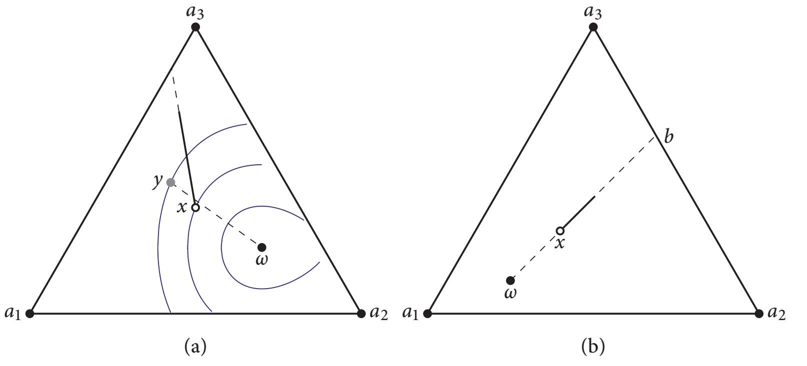

Figure 8: Left: Any convex set containing the most preferred point and a point "behind" the budget intersects the hyperplane which contains the budget. Right: The revealed worse set consists merely of the line $\overline{xb}$.

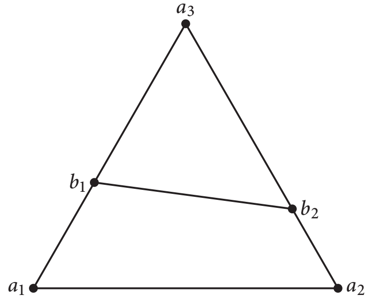

Figure 9: A budget as suggested in the text.

with $\tilde{g} : \mathbb{R}_+^{n-1} \to \mathbb{R}$ being a continuous and linear function. To see this, we switch from our usual presentation of $D(A)$ to the so called Marschak-Machina triangle (Marschak 1950, Machina 1982): for n outcomes, we have that $p_n = 1 - \sum_{i=1}^{n-1} p_i$, so all lotteries can be represented by a lottery in the $(p_1, \ldots, p_{n-1})$ plane. See Figure 11 for an example and Section A.1 for a way to construct a function g from the lotteries in B.

Because $g(x) = 0$ for all x on a budget, we have that both $\tilde{g}(x)$ and $-\tilde{g}(x)$ describe the same budget. However, we need to have $\tilde{g}(x) < 0$ if x is on the "worse side" of the budget, i.e. if $x \in sdRW(x)$, and $\tilde{g}(x) > 0$ if x is not on the "worse side" of a budget, i.e. if $x \notin dRW(x)$. Furthermore, because we assume that ω is the unique maximal element, we need $\tilde{g}(x) < 0$ for all x which are not on a budget if ω is an element of the budget. We therefore define the function $g(x) : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as

$$g(x) = \begin{cases} \tilde{g}(x) & \text{if } \tilde{g}(\omega) > 0, \\ -\tilde{g}(x) & \text{if } \tilde{g}(\omega) < 0, \\ -|\tilde{g}(x)| & \text{if } \tilde{g}(\omega) = 0. \end{cases} \tag{6}$$

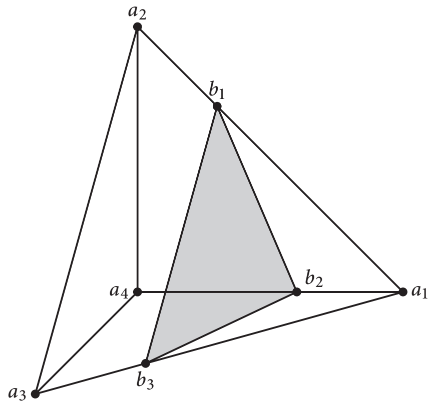

Figure 10: A budget in a 3-simplex

See Figure 12 for an example. One immediate advantage of this characterization of budgets is that we do not need to check whether a given lottery is in the convex hull of a set of lotteries to determine on which side of a budget that lottery is. Given a function $g_i(x)$ which describes the ith budget in $D(A)$, budgets are then defined as[3]

$$B^i = \left\{x \in \mathbb{R}_+^{n-1} : g_i(x) = 0\right\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \leq 1\right\}. \tag{7}$$

3.5 *The Revealed Preference Relation*

Given the definitions of budgets and the analysis in Section 3.4 and Section 3.2, we can now define the revealed preference relation $R_A \subseteq D(A) \times D(A)$ as

$$x^i R_A x^j \text{ if } g_i(x^j) \leq 0 \tag{8}$$

and its transitive closure, i.e. the smallest transitive relation on $D(A)$ that contains R, is denoted R_A^*. Furthermore, we define the strictly revealed preference relation $P \subseteq D(A) \times D(A)$ as

$$x^i P_A x^j \text{ if } g_i(x^j) < 0 \tag{9}$$

and its transitive close is denoted P^*.

4 REPRESENTATION

4.1 *Axioms: Refutable Conditions*

Given our construction of the revealed worse set and the revealed preference relation R, can we find refutable conditions for the hypothesis of quasiconcave preferences? Necessary

[3] We drop the subscript A for budgets.

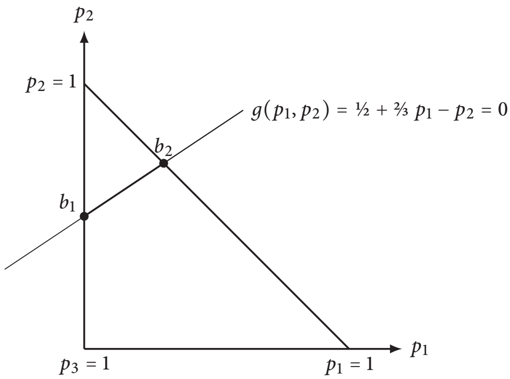

Figure 11: The Marschak-Machina triangle with $b_1 = (0, \frac{1}{2}, \frac{1}{2})$ and $b_2 = (\frac{3}{10}, \frac{7}{10}, 0)$.

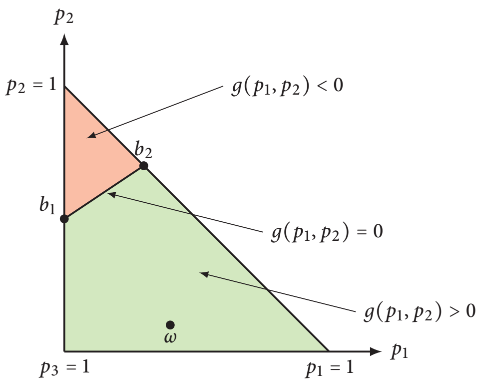

Figure 12: The Marschak-Machina triangle: Regions for $g(x) \lesseqgtr 0$.

conditions are easily found; however, as will be shown, we can also find conditions which are necessary and sufficient for the existence of a quasiconcave utility function that rationalizes the observations.

Because of the assumption that ω is the unique maximal element in $D(A)$, we need to augment any set of oberservation by an observation ω – otherwise, we might have that for some observation i, $g_i(\omega) = 0$ and $x^i \neq \omega$, and as will be seen later, this is not a violation of the Generalized Axiom. We will therefore augment any set of m observations by an observation consisting of $C_A(B^{m+1}) = \omega$ and a function g_{m+1}, which is defined as

$$g_{m+1} = -d(x, \omega), \tag{10}$$

where d is the Euclidean distance function, so $g_{m+1}(x) < 0$ for all $x \neq \omega$.

Let $M = \{1, \ldots, m+1\}$. We can now state our axioms.

Definition 1. *We say a set of data* $S_A = \{x^i, B^i\}_{i \in M}$ *for different budgets* B_i *on a single* $D(A)$ *satisfies the* Weak Axiom of Revealed Quasiconcave Preference (WARQ) *if for all* $\{i, j\} \subseteq M$ *such that* $x^i \neq x^j$

$$x^i R_A x^j \;\; implies \;\; g_j(x^i) > 0. \tag{11}$$

We say a set of data satisfies the Strong Axiom of Revealed Quasiconcave Preference (SARQ) *if for all* $\{i, j\} \subseteq M$ *such that* $x^i \neq x^j$

$$x^i R^*_A x^j \;\; implies \;\; g_j(x^i) > 0. \tag{12}$$

It should be obvious that WARQ implies SARQ but not vice versa, and that a violation of WARQ or SARQ implies that preferences cannot be quasiconcave; see Figures 13.(a) and 13.(b) for examples.

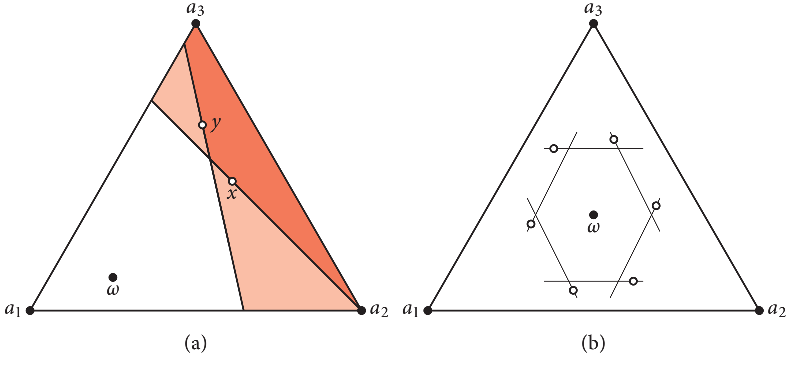

Figure 13: Left: A violation of WARQ and SARQ. Right: A violation of SARQ but not of WARQ.

Definition 2. *We say a set of data satisfies the* Generalized Axiom of Revealed Quasiconcave Preference (GARQ) *if for all* $\{i, j\} \subseteq M$

$$x^i R^*_A x^j \;\; implies \;\; g_j(x^i) \geq 0. \tag{13}$$

4.2 *Representation*

We will now proceed to show that our Generalized Axiom (Strong Axiom) is a necessary and sufficient condition for rationalizability or representation of the data by a continuous and (strictly) quasiconcave utility function. Because these two axioms are fairly easy to test in practice given a finite set of observations, they offer an efficient way to refute the hypothesis of (strictly) quasiconcave preferences.

While an indirect proof of existence is feasible and can be illustrated intuitively, we will only sketch this approach briefly in Remark 2. We present a constructive proof based on a generalization of Afriat's (1967) theorem due to Forges and Minelli (2009).[4]

[4] See also the clarification of Afriat's result by Diewert (1973).

For the representation theorems, we will continue to assume that budgets are constructed from points on the boundary of $D(A)$.

Definition 3. *We say a function $U(x)$* rationalizes *a set of observations $S_A = \{x^i, B^i\}_{i \in M}$ if*

$$U(x^i) \geq U(x) \;\; \textit{if} \;\; x^i R_A x \tag{14}$$

for all $i \in M$.

Note that if we can find a utility function on $D(A)$ which rationalizes the data in $D(A)$, we can easily find a utility function on the $(p_1, \ldots, p_{n-1})$ plane which has the same properties and rationalizes the data in the $(p_1, \ldots, p_{n-1})$ plane because $D(A)$ and the Marschak-Machina triangle are order isomorphic.

Theorem 1. *In the $(p_1, \ldots, p_{n-1})$ plane, let budgets be given by $B_i = \{x \in \mathbb{R}_+^{n-1} : g_i(x) = 0\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \leq 1\right\}$ with $g_i : \mathbb{R}_+^{n-1} \to \mathbb{R}$ being a continuous and linear function and $g_i(x^i) = 0$ for all $i \in M$. The following conditions are equivalent:*

(i) The set of observations $S_A = \{x^i, B_i\}_{i \in M}$ satisfies GARQ.

(ii) There exists a function $U : D(A) \to \mathbb{R}$ that is satiated at ω, nonsatiated at every other lottery in $D(A)$, continuous and quasiconcave on int$D(A)$ and rationalizes the set of observations S_A.

(iii) There exist numbers $\{\phi_i, \lambda_i\}_{i \in M}$, $\lambda_i > 0$, such that for all $i, j \subseteq M$

$$\phi_j \leq \phi_i + \lambda_i \, g_i(x^j). \tag{15}$$

Proof We proceed to show (ii) ⇒ (i), (i) ⇒ (iii), and finally (iii) ⇒ (ii) by contruction of an actual utility function which rationalizes the data.

Proof of (ii) ⇒ (i): Let $U(x)$ rationalize the data. If $x^i R_A x^j$, then $U(x^i) \geq U(x^j)$; if $x^i R_A^* x^j$, then there exist indices $(k, \ldots, \ell)$ such that $x^i R_A x^k R_A \ldots R_A x^\ell R_A x^j$, and $U(x^i) \geq U(x^k) \geq \ldots \geq U(x^j)$ implies $U(x^i) \geq U(x^j)$. We want to show that this implies $g_j(x^i) \geq 0$. Suppose first that $x^i \neq \omega$. If $g_j(x^i) < 0$, by the nonsatiation of U we can find an $x \in D(A)$ such that $g_j(x) < 0$ and $U(x) > U(x^i) \geq U(x^j)$. But then U does not rationalize the data. Suppose instead that $x^i = \omega$. Then $g_j(\omega) < 0$ is ruled out by the definition of g (Definition 6).

Proof of (i) ⇒ (iii): We need the following definition: A square matrix $\Gamma = [\gamma_{ij}]_{(m+1)\times(m+1)}$ is cyclically consistent if $\gamma_{ii} = 0$ for every $i \in M$ and for every chain $\{j, k, \ell, \ldots, c\} \subset M$, $\gamma_{jk} \leq 0$, $\gamma_{k\ell} \leq 0$, …, $\gamma_{cj} \leq 0$ implies that all terms are zero (see Forges and Minelli 2009, Section 1.2). Then the following lemma can be shown to hold.

Lemma 1. *If a square matrix $\Gamma = [\gamma_{ij}]_{(m+1)\times(m+1)}$ is cyclically consistent, there exist numbers $\{\phi_i, \lambda_i\}_{i \in M}$, $\lambda_i > 0$, such that for all $\{i, j\} \subseteq M$ we have*

$$\phi_j \leq \phi_i + \lambda_i \gamma_{ij}. \tag{16}$$

For the proof see Fostel et al. (2004), Sections 2 and 3.

Construct the matrix Γ such that $\gamma_{ij} = g_i(x^j)$. It can then be shown that if S_A satisfies GARQ, Γ is cyclically consistent:

Lemma 2. *A square matrix $\Gamma = [\gamma_{ij}]_{(m+1)\times(m+1)}$ with $\gamma_{ij} = g_i(x^j)$ is cyclically consistent if and only if S_A satisfies GARQ.*

The same proof as in Forges and Minelli (2009), Section 1.2, can be applied. With Lemma 1 and 2 it follows that (i) ⇒ (iii).

Proof of (iii) ⇒ (ii): Define $V : \mathbb{R}_+^{n-1} \to \mathbb{R}$ by

$$V(x) = \min_{i\in M} \{\phi_i + \lambda_i\, g_i(x)\}\,. \tag{17}$$

As the minimum of finitely many concave and continuous functions, $V(x)$ is concave (and therefore quasiconcave) and continuous.

To show that it rationalizes the data, note that for all $j \in M$ we have $V(x^j) = \phi_j$. To see this, let $K = \left\{\arg\min_{i\in M}\{\phi_i + \lambda_i\, g_i(x^j)\}\right\}$. If $j \notin K$, then by (15) we have $\phi_j < \phi_k + \lambda_k\, g_k(x^j) = \min_{i\in M}\{\phi_i + \lambda_i\, g_i(x^j)\} = V(x^j)$. But since $V(x^j) = \min_{i\in M}\{\phi_i + \lambda_i\, g_i(x^j)\} \le \phi_j + \lambda_j\, g_j(x^j) = \phi_j$, we have $\phi_j < V(x^j) \le \phi_j$, a contradiction. For any x such that $g_j(x) \le 0$ (i.e. $x^j R_A x$) we have $V(x) \le \phi_j + \lambda_j\, g_j(x) \le \phi_j = V(x^j)$ and for any x such that $g_j(x) < 0$ (i.e. $x^j P_A x$) we have $V(x) < \phi_j + \lambda_j\, g_j(x) \le \phi_j = V(x^j)$. Finally, we have $V(\omega) = \phi_{m+1}$ because $\omega = x^{m+1}$, and for all $x \in D(A)$, $x \ne \omega$, we have $V(x) < \phi_{m+1}$. To see this, note that $\phi_{m+1} + \lambda_{m+1}\, g_{m+1}(x) < \phi_{m+1}$ by the definition of $g_{m+1}(x)$ in Eq. (10), so $\min_{i\in M}\{\phi_i + \lambda_i\, g_i(x)\} < \phi_{m+1}$. ■

Remark 1. *In Theorem 1, (i) ⇒ (iii) can also be shown by an algorithm very much like Varian's (1982) algorithm. Such an algorithm is given in the appendix (A.2, Algorithm 2 and Lemma 3).*

Theorem 2. *In the $(p_1,\ldots,p_{n-1})$ plane, let budgets be given by $B^i = \{x \in \mathbb{R}_+^{n-1} : g^i(x) = 0\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \le 1\right\}$ with $g^i : \mathbb{R}_+^{n-1} \to \mathbb{R}$ being a continuous and linear function and $g_i(x^i) = 0$ for all $i \in M$. The following conditions are equivalent:*

(i) The set of observations $S_A = \{x^i, B^i\}_{i\in M}$ satisfies SARQ.

(ii) There exists a function $U : D(A) \to \mathbb{R}$ that is satiated at ω, nonsatiated at every other lottery in $D(A)$, continuous and strictly quasiconcave on int$D(A)$ and rationalizes the set of observations S_A.

(iii) There exist numbers $\{\phi_i, \lambda_i\}_{i\in M}$, $\lambda_i > 0$, such that for all $i, j \subseteq M$

$$\phi_j < \phi_i + \lambda_i\, g_i(x^j) \qquad \textit{for all } \{i,j\} \subseteq M \textit{ with } x^i \ne x^j, \tag{18a}$$

$$\phi_j = \phi_i \qquad \textit{for all } \{i,j\} \subseteq M \textit{ with } x^i = x^j. \tag{18b}$$

Proof We proceed in the same way as in the proof to Theorem 1.

Proof of (ii) ⇒ (i): Let $U(x)$ rationalize the data. If $x^i R_A x^j$, then $U(x^i) \ge U(x^j)$;

if $x^i R_A^* x^j$, then there exist indices $(k, \ldots, \ell)$ such that $x^i R_A x^k R_A \ldots R_A x^\ell R_A x^j$, and $U(x^i) \geq U(x^k) \geq \ldots \geq U(x^j)$ implies $U(x^i) \geq U(x^j)$. We want to show that this implies $g_j(x^i) > 0$. If $g_j(x^i) < 0$, by the nonsatiation of U we can find an $x \in D(A)$ such that $g_j(x) < 0$ and $U(x) > U(x^i) \geq U(x^j)$. But then U does not rationalize the data. If $g_j(x^i) = 0$, then by strict quasiconcavity of U we have that for $y = \lambda\, x^i + [1-\lambda]\, x^j$, $U(y) > \max\{U(x^i), U(x^j)\}$ so $U(z) > U(x^i)$. But $g_j(y) = 0$, which implies $U(x^j) \geq U(z)$, so $U(x^i) \geq U(x^j) \geq U(z) > U(x^i)$, a contradiction.

Proof of (i) ⇒ (iii): This can either be shown using a Theorem of the Alternative (Rockafellar 1970, Theorem 22.2, pp.198–199) in analogy to Matzkin and Richter (1991, Lemma 1) by replacing $\alpha_{ij} = p^i(x^j - x^i)$ in their paper with $g_i(x^j)$, or by means of an algorithm as in Varian (1982, Algorithm 3), which is done in the Appendix (A.3) with Lemma 4.

Proof of (iii) ⇒ (ii): We follow Matzkin and Richter (1991) in constructing the utility function. Let $T > 0$ and define $f : \mathbb{R}_+^{n-1} \to \mathbb{R}$ by

$$f(x_1, \ldots, x_{n-1}) = \left[\sum_{i=1}^{n-1} (x_1)^2 + T\right]^{½} - T^{½}. \tag{19}$$

There exists an $\varepsilon_0 > 0$ such that $\phi_j < \phi_i + \lambda_i\, g_i(x^j) - \varepsilon_0$ for all $\{i, j\} \subseteq M$ with $x^i \neq x^j$ and the other two conditions of Theorem 2 (iii) hold as well, as can also be seen in the proof of Lemma 4. Then we can choose an ε so small that

$$\phi_j < \phi_i + \lambda_i\, g_i(x^j) - \varepsilon\, f(x^j - x^i) \qquad \text{for all } \{i, j\} \subseteq M \text{ with } x^i \neq x^j, \tag{20a}$$

$$\lambda_i > 0 \qquad \text{for all } i \in M, \tag{20b}$$

$$\phi_i = \phi_j \qquad \text{for all } \{i, j\} \subseteq M \text{ with } x^i = x^j. \tag{20c}$$

For each $i \in M$ we define $\pi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\pi_i(x) \equiv \phi_i + \lambda_i\, g_i(x) - \varepsilon\, f(x - x^i). \tag{21}$$

Clearly, f is strictly convex, so each π_i is strictly concave. Furthermore, $\pi_i(x^i) = \phi_i$, because $f(x) = 0 \Leftrightarrow x = \mathbf{0}$. Now define $V : \mathbb{R}_+^{n-1} \to \mathbb{R}$ by

$$V(x) = \min_{i \in M} \{\pi_i(x)\}. \tag{22}$$

As the minimum of finitely many strictly concave and continuous functions, $V(x)$ is strictly concave (and therefore strictly quasiconcave) and continuous.

To show that it rationalizes the data, note that for all $j \in M$ we have $V(x^j) = \phi_j$. To see this, let $K = \{\arg\min_{i \in M}\{\pi_i(x^j)\}\}$. If $j \notin K$, then by (18a) we have $\phi_j < \pi_k(x^j) = \min_{i \in M}\{\pi_(x^j)\} = V(x^j)$. But since $V(x^j) = \min_{i \in M}\{\pi_i(x^j)\} \leq \pi_j(x^j) = \phi_j$, we have $\phi_j < V(x^j) \leq \phi_j$, a contradiction. For any x such that $g_j(x) \leq 0$ (i.e. $x^j R_A x$) we have $V(x) < \pi_j(x) \leq \phi_j = V(x^j)$ and obviously for any x such that $g_j(x) < 0$ (i.e. $x^j P_A x$) we have $V(x) < \pi_j(x) \leq \phi_j = V(x^j)$. Finally, we have $V(\omega) = \phi_{m+1}$ because $\omega = x^{m+1}$, and for all $x \in D(A)$, $x \neq \omega$, we have $V(x) < \phi_{m+1}$. To see this, note that $\pi_{m+1}(x) < \phi_{m+1}$ by the definition of $g_{m+1}(x)$ in Eq. (10), so $\min_{i \in M}\{\pi_i(x)\} < \phi_{m+1}$. ∎

Remark 2. *The equivalence between (i) and (ii) in Theorem 1 and 2 can be shown indirectly in analogy to Stigum (1973) and using a result due to Debreu (1959, p. 56, Theorem 4.6.1). Intuitively speaking, the idea is that if SARQ is satisfied we have "enough space" to "draw" indifference curves of a strictly quasiconcave preference. To show this it would be necessary to establish that we can find sets* $L_A^+(x) = \{p \in D(A) : pQx\}$ *and* $L_A^-(x) = \{p \in D(A) : xQp\}$*, where* Q*,* $R^* \subseteq Q \subset D(A) \times D(A)$*, is a complete binary relation, such that for the revealed preferred set* $RP_A(x)$ *(defined in Section 5.1) we have* $RP_A(x) \subseteq \overline{L_A^+(x)}$*, and for the revealed worse set* $RW_A(x)$ *we have* $RW_A(x) \subseteq L_A^-(x)$*, where* $\overline{L_A}$ *denotes the closure of the sets. These sets would then satisfy* $\overline{L_A^+(x)} \cap L_A^-(x) = \emptyset$*,* $\overline{L_A^+(x)} \cup L_A^-(x) = L_A^+(x) \cup \overline{L_A^-(x)} = D(A)$*, and* $L_A^+(x)$ *would be strictly convex.*

Remark 3. *It has been shown that the Weak Axiom of Revealed Preference (for the common commodity space* $\mathbb{R}_+^m$*) implies the Strong Axiom of Revealed Preference for the case of two commodities but not for more (see Peters and Wakker 1994, John 1997 and Heufer 2007). Analogously to the proof in Heufer (2007), one can show that WARQ implies SARQ for three outcomes when the most preferred lottery* ω *is on the boundary of the probability simplex. In all other cases (*ω *in the interior, more than three outcomes), WARQ does not imply SARQ. A loose version of the argument is this: In Figure 13.(b) SARQ but not WARQ is violated. In this figure,* ω *is in the interior of the simplex. Note that the preference cycle forms a closed curve around* ω*. Suppose now that* ω *is on the boundary of the simplex. In that case, no preference cycle can be closed around* ω*. Now suppose there are more than three outcomes. In this case, a preference cycle does not need to be a closed curve around* ω*; it could indeed be a closed curve on one "side" of* ω*.*

5 FURTHER ANALYSIS

5.1 *Recoverability: Revealed Worse and Preferred Sets of Arbitray Lotteries*

It was shown in Section 3.2 how the revealed worse set can be constructed under the hypothesis of quasiconcave preferences. In this section, it is shown how one can construct the revealed worse and the revealed preferred set of arbitrary lotteries in $D(A)$ which were not observed as decisions. The analysis here closely follows Varian's (1982) approach. As in his work, we will focus on quasiconcavity and consistency with GARQ.

Definition 4. *Given any lottery* $x^0 \in D(A)$ *not previously observed we define the set of budgets in* $D(A)$ *which* support x^0 *by*

$$\Theta_A(x^0) = \left\{B^0 : \{B^i, x^i\}_{i \in M \cup \{0\}} \text{ satisfies GARQ and } g_0(x^0) = 0\right\}, \tag{23}$$

where

$$B^0 = \{x \in \mathbb{R}_+^{n-1} : g_0(x) = 0\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \leq 1\right\}.$$

Note that Theorem 1 implies $\Theta_A(x^0)$ is nonempty for all x^0. Given $\Theta_A(x^0)$ we can easily describe the set of all lotteries revealed worse than x^0: We require that for every x in the revealed worse set of x^0, we have that $x^0 P_A^* x$ holds for all budgets in $\Theta_A(x^0)$ (e.g. if $x^0 P_A^* x$ according to some $B^0 \in \Theta_A(x^0)$ but not according to some other $B^0 \in \Theta_A(x^0)$ then x is not in the revealed worse set). More succinctly, we define the revealed worse set of x^0, $RW_A(x^0$, by

$$RW_A(x^0) = \left\{x \in D(A) : x^0 P_A^* x \text{ for all } B^0 \in \Theta_A(x^0)\right\}. \tag{24}$$

Similarly, we can define the revealed preferred set of x^0, $RP_A(x^0)$, by

$$RP_A(x^0) = \left\{x \in D(A) : x P_A^* x^0 \text{ for all } B \in \Theta_A(x)\right\}. \tag{25}$$

Figure 14 shows two examples.

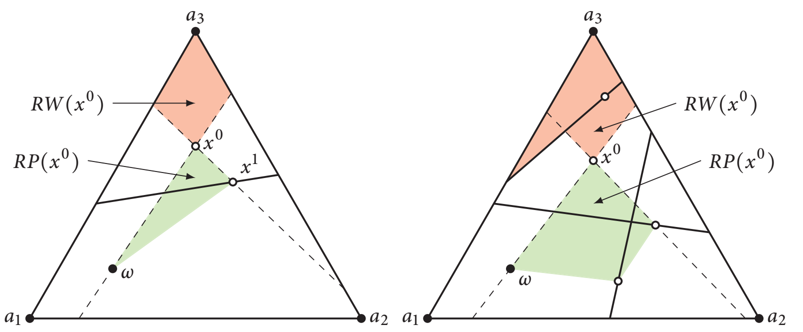

Figure 14: Revealed worse and revealed preferred sets of an unobserved point x^0.

From Figure 14 it appears that $RP_A(x^0)$ is just the convex hull of all x^i which are revealed preferred to x^0. In fact, define

$$CM_A(x^0) = \text{interior of conv}(\{x \in \{x^i\}_{i \in M} : x R_A^* x^0\}), \tag{26}$$

and let $\overline{CM}_A(x^0)$ be the closure of $CM_A(x^0)$ and

$$\text{conv}(\{x^i\}_{i=1}^{\ell}) = \left\{\sum_{i=1}^{\ell} \lambda_i x^i : \lambda_i \in [0,1], \sum_{i=1}^{\ell} \lambda_i = 1\right\} \tag{27}$$

is the convex hull of a set of points. Then the following can be shown to hold:

Proposition 3. *Let $S_A = \{x^i, B^i\}_{i \in M}$ be a set of observations and let $RP_A(x^0)$ be defined by (25). Then*

$$CM_A(x^0) \subseteq RP_A(x^0) \subseteq \overline{CM}_A(x^0).$$

Proof In analogy to Varian (1982, Fact 12, p. 960) and Knoblauch (1992, Proposition 1, p. 661). ■

Because it is easy to check whether a point is in the convex hull of a set of points and to determine whether a point on the boundary of $RP_A(x^0)$ belongs to $RP_A(x^0)$, Proposition 5.1 completely describes $RP_A(x^0)$. Furthermore, from Varian (1982, Fact 3, p. 951) we know that x^0 is in $RW_A(x)$ if and only if x is in $RP_A(x^0)$, so we can easily determine whether or not a point x is in either $RW_A(x^0)$ or $RP_A(x^0)$. See Section A.4 for an efficient way to determine whether a point is in the convex hull of a set of points.

Notice that Figure 14 provides information about the possible indifference curves passing through x^0. Any such indifference curve cannot intersect $RP_A(x^0)$ or $RW_A(x^0)$, that is, the set of lotteries preferred to x^0 must contain $RP_A(x^0)$, and must be contained in the complement of $RW_A(x^0)$, which is the basis for the idea expressed in Remark 2.

5.2 *General Budget Sets*

Suppose that instead of being $(n-1)$-dimensional hyperplanes, budgets are more general sets in $D(A)$. Can we still find refutable conditions and contruct quasiconcave utility functions? One could, in principle, use the generalization of Afriat's Theorem due to Forges and Minelli (2009), using continuous, differentiable, and quasiconvex functions $g(x)$ describing the budget, as in Figure 15. The problem, however, is that Forges and Minelli exploit monotonicity of preferences on the common commodity space $\mathbb{R}^{\ell}$: their budgets are bounded by curves which "bulge away" from the origin. To transfer their analysis to the probability simplex, we would need to define the convexity as "curved towards" ω.

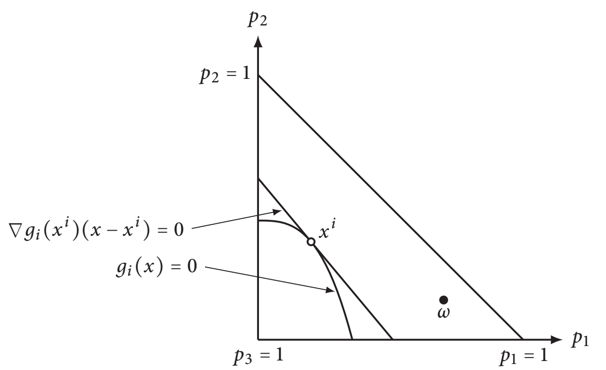

Figure 15: A quasiconvex budget.

If the functions $\tilde{g}_i(x)$ satisfy differentiablity and are convex in the sense that they are "curved towards" ω, we can indeed use Forges and Minelli's analysis and define

$$C^i = \left\{x \in \mathbb{R}_+^{n-1} : \nabla g_i(x^i)(x - x^i) = 0\right\} \cap \left\{x \in \mathbb{R}_+^{n-1} : \sum x_j \leq 1\right\} \tag{28}$$

and test the "linearized" set of data $\bar{S}_A = \{x^i, C^i\}_{i \in M}$ for consistency with GARQ or SARQ. A similar approach could be used for even more general budget sets such as the one depicted in Figure 16.

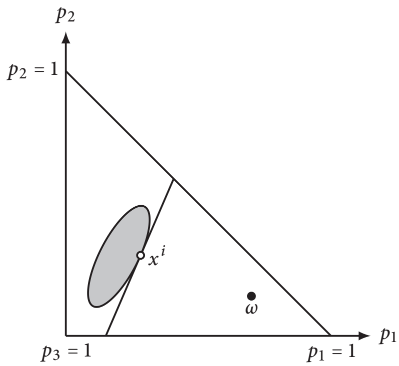

Figure 16: A strictly convex budget set.

5.3 *What to do When the Most Preferred Lottery is Unknown*

Suppose we do not know the most preferred lottery in $D(A)$, i.e. the point ω. In principle, GARQ and SARQ are still testable conditions: if an axiom is satisfied for *some* arbitrary $\omega \in D(A)$, we cannot reject the hypothesis of quasiconcavity of preferences. The problem is then to find an efficient way to test if there is such an ω. One efficient way is to let $\Omega = \{x^i\}_{i=1}^m$ and test the m sets $\{x^i, B^i\}_{i=1}^m \cup \{\Omega^j, B^{\Omega^j}\}$ for $j \in \{1, \ldots, m\}$, where (with an abuse of notation) B^{Ω^j} is the budget for Ω^j defined by a function as in Eq. (10). If this data set satisfies GARQ or SARQ, then we can safely conclude that the hypothesis of quasiconcavity of preferences cannot be rejected. However, as Figure 17 shows, if GARQ is rejected we can still not reject our hypothesis.

6 EXAMPLES AND THE POWER OF THE TEST

6.1 *Examples*

Figure 18 shows a set of budgets which will be used to illustrate the revealed preferred and revealed worse sets, and the utility function constructed in the proof of the theorem. The most preferred lottery is $\omega = (⅓, ⅓)$.

Figure 18 also shows decisions generated by the utility function

$$V(p_1, p_2) = -\left[\sum_{i=1}^{3} (⅓ - p_i)^2\right]^{½}$$

with $p_3 = 1 - p_1 - p_2$. The smooth closed curve around ω is an indifference curve of that utility function for a lottery close to ω which was not observed as a choice; it is contained in

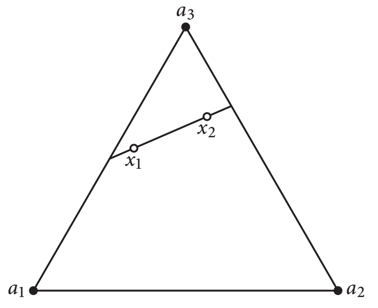

Figure 17: GARQ is clearly satisfied, but the test described in Section 5.3 will not confirm consistency with GARQ.

the indifference curve of the utility function constructed by using the numbers computed by Algorithm 3.

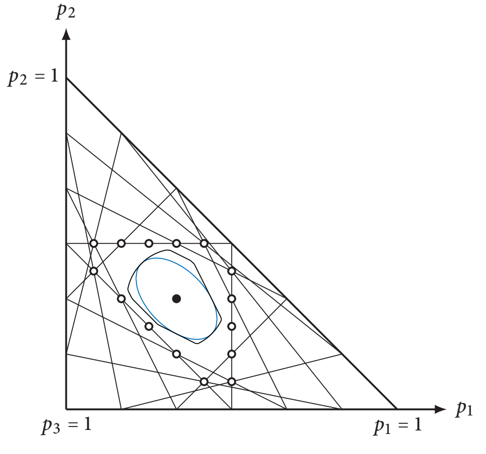

Figure 18: Choices generated by a specified utility function.

Figures 19 and 20 show random decisions on the same budgets; the budgets are not drawn here. The indifference curve of the constructed utility function through one of the decisions and the revealed worse and revealed preferred set of that choice are shown. Notice that, as pointed out in Section 5.1, the indifference curve contains the revealed preferred set, and is contained in the complement of the revealed worse set.

6.2 *Power of the Test*

Bronars (1987) suggested a Monte Carlo approach to approximate the power a revealed preference test has against random behavior. The question is this: Given a set of budgets,

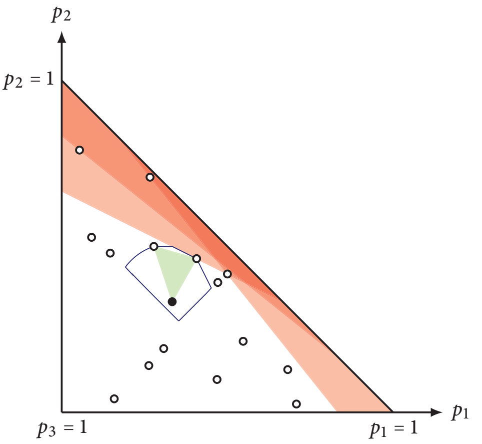

Figure 19: Random choices which satisfy SARQ.

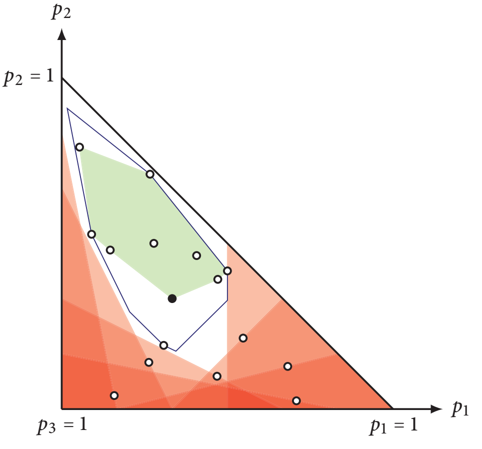

Figure 20: Random choices which satisfy SARQ.

what is the probability that a consumer who makes random decisions still satisfies a revealed preference axiom? Obviously, if this probability is high, a test which confirms consistency of a set of observations with an axiom is not as informative as we would like it to be: while we cannot reject our hypothesis, it may very well be the case that the observations were not generated by optimizing behavior. Bronars' idea was to generate random choices on a set of budgets, repeat this many times, and compute the fraction of random sets which do not pass the test. This then is the approximate power.

For the fifteen budget depicted in Figure 18 we simulated choices by drawing fifteen scalars λ_i from a uniform distribution on the unit interval. We let $x^i = \lambda\, b_1^i + [1-\lambda]\, b_2^i$ for $i \in \{1, \ldots, 15\}$ and $x^{16} = \omega = (⅓, ⅓)$.

Because all budgets are distinct, SARQ and GARQ are equivalent. We generated 200000 random choice sets. Of these 200000 random sets, only 1696 sets satisfied the axioms, yielding a very high power of 99.152%. This was repeated by drawing three, six, nine, and twelve out of the 15 budgets at random each time. Table I shows the results.

Number of Budgets	3	6	9	12	15
Power	12.917%	47.760%	75.994%	91.320%	99.152%

Table I: Power of the test depending on the number of budgets used.

7 Discussion and Conclusions

This paper introduced a nonparametric approach to the analysis of decisions on a probability simplex. Easily testable necessary and sufficient conditions were found which guarantee the existence of a quasiconcave utility function which rationalizes a set of observations. It was shown how one can construct an actual utility function, and how one can recover preferences. The analysis is much in the spirit of Afriat's (1967) and Varian's (1982) contribution to revealed preference and nonparametric demand analysis.

While the approach described here is in principle well suited for a laboratory experiment, there are practical issues which need to be addressed. First, unless the recruited subjects are students of fields likely to cover simplices, it would probably be impractical to attempt to explain subjects even what a probability simplex is. Second, the presentation of a hyperplane inside a tetrahedron would require subjects to choose at least two variables to determine a point on the hyperplane, and it is not clear in how far subjects would be aware of what they are doing. Also, a graphical presentation for lotteries over more than four outcomes might be difficult if not impossible.

As for the second point, an experimental investigation of choice behavior should perhaps be restricted to three outcomes. The first point was already solved elegantly by Sopher and Narramore (2000) and Karni et al. (2008): Subjects were presented a slider on a computer screen which they used to determine the λ for the optimal combination of the two extreme lotteries b_1 and b_2, i.e. they could choose $\lambda\, b_1 + (1-\lambda)\, b_2$ with a simple mechanism. Their options were presented by a pie chart – a concept most subjects are probably familiar with.

A APPENDIX

A.1 *Constructing the Implicit Budget Function*

A function $\tilde{g}_i$ such that $\tilde{g}_i(x) = 0$ if and only if $x \in \tilde{B}^i$ can be easily found by solving a linear programming problem. Let $\mathbf{b}^i$ be the $(n-1) \times (n-1)$ matrix that describes the budget in Marschak-Machina triangle, i.e. $\mathbf{b}^i_{jk}$ is the p_k coordinate of the point $\mathbf{b}^i_j$ which is a vertex of the budget. Let α be a scalar and $\beta = (\beta_1, \ldots, \beta_{n-1})$. Then solve

$$\begin{aligned} h^* = & \min_{\alpha\in\mathbb{R},\beta\in\mathbb{R}^{n-1}} \quad 0\cdot\alpha + 0\cdot\beta \\ & \text{subject to} \quad \alpha + \sum_{k=1}^{n-1}(\beta_k\,\mathbf{b}^i_{jk}) = 0 \;\; \text{for all} \;\; j \in \{1,\ldots,n-1\} \\ & \qquad\qquad\quad \alpha + \sum_{j=1}^{n-1}\beta_j = 1 \end{aligned} \tag{29}$$

and let $\tilde{g}_i(x) = \alpha + \sum_{k=1}^{n-1}(\beta_k\, x_k)$.

A.2 *An Algorithm to Construct the Numbers in Theorem 1*

This is a straightforward adaptation of Varian's (1982) algorithm to construct the Afriat numbers. We also need an algorithm which finds a maximal element of a binary relation Q, called MaxElement(I, Q), where $I = \{1, \ldots, m\}$ is a set of indices. An element x^μ of a set $\{x^i\}_{i=1}^m$ is maximal with respect to a binary relation Q if $x^i Q x^\mu$ implies $x^\mu Q x^i$. We can use Algorithm 2 in Varian (1982):

Algorithm 1.
Input: *A reflexive and transitive binary relation Q defined on a finite set $\{x^i\}_{i\in M}$ indexed by $I = \{1, \ldots, m+1\}$.*
Output: *An index μ where $x^i Q x^\mu$ implies $x^\mu Q x^i$.*

1. *Set $\mu = 1$ and $q^0 = x^1$.*
2. *For each $i \in M$, if $x^i Q q^{i-1}$ set $q^i = x^i$ and $\mu = i$. Otherwise set $q^i = q^{i-1}$.*

This algorithm correctly computes a maximal element (see Varian 1982, Fact 15).

Algorithm 2.
Input: *A set of observations $\{x^i\}_{i\in M}$ and $\{g_i(x)\}_{i\in M}$ and the relation R^*_A that satisfies GARQ.*
Output: *A set of numbers $\{\phi_i\}_{i\in M}$ and $\{\lambda_i\}_{i\in M}$.*

1. *Set $I = \{1, \ldots, m+1\}$, $B = \varnothing$.*
2. *Let $\mu = MaxElement(I, R^*_A)$.*
3. *Set $E = \{i \in I : x^i R^*_A x^\mu\}$. If $B = \varnothing$, set $\phi_\mu = \lambda_\mu = 1$ and go to Step 6. Otherwise go to Step 4.*
4. *Set $\phi_\mu = \min_{i\in E} \min_{j\in B} \min\{\phi_j + \lambda_j\, g_j(x^i), \phi_j\}$.*
5. *Set $\lambda_\mu = \max_{i\in E} \max_{j\in B} \max\{(\phi_j - \phi_\mu)/g_i(x^j), 1\}$.*
6. *For all $i \in E$, set $\phi_i = \phi_\mu$ and $\lambda_i = \lambda_\mu$.*

7. *Set* $I = I \backslash E$, $B = B \cup E$. *If* $I = \varnothing$, *stop. Otherwise, go to Step 2.*

Lemma 3. *Algorithm 2 computes* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$ *which satisfy the inequalities in Theorem 1, condition (iii).*

Proof Identical to the proof in Varian (1982), except that $p^i(x^j - x^i)$ is replaced by $g_i(x^j)$. ■

A.3 *An Algorithm to Construct the Numbers in Theorem 2*

Again, this is an adaptation of Varian's (1982) algorithm to construct the Afriat numbers, using additional ideas from Chiappori and Rochet (1987) and Matzkin and Richter (1991).

Algorithm 3.
Input*: A set of observations* $\{x^i\}_{i\in M}$ *and* $\{g_i(x)\}_{i\in M}$ *and the relation* R_A^* *that satisfies SARQ.*
Output*: A set of numbers* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$*.*

1. *Set* $I = \{1, \ldots, m+1\}$, $B = \varnothing$, *and choose an* $\varepsilon > 0$.
2. *Let* $\mu = MaxElement(I, R_A^*)$.
3. *Set* $E = \{i \in I : x^i R_A^* x^\mu\}$. *If* $B = \varnothing$, *set* $\phi_\mu = \lambda_\mu = 1$ *and go to Step 6. Otherwise go to Step 4.*
4. *Set* $\phi_\mu = \min_{i\in E} \min_{j\in B} \min\{\phi_j + \lambda_j\, g_j(x^i) - \varepsilon, \phi_j - \varepsilon\}$.
5. *Set* $\lambda_\mu = \max_{i\in E} \max_{j\in B} \max\{(\phi_j - \phi_\mu + \varepsilon)/g_i(x^j), 1\}$.
6. *For all* $i \in E$, *set* $\phi_i = \phi_\mu$ *and* $\lambda_i = \lambda_\mu$.
7. *Set* $I = I \backslash E$, $B = B \cup E$. *If* $I = \varnothing$, *stop. Otherwise, go to Step 2.*

Lemma 4. *Algorithm 3 computes* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$ *which satisfy the inequalities in Theorem 2, condition (iii).*

Proof We need to show the following:

(a) $\phi_i = \phi_j$ for all $j \in B$ and $i \in E$ such that $x^i = x^j$,

(b) $\phi_i = \phi_j$ for all $\{i, j\} \subseteq E$ such that $x^i = x^j$.

(c) $\phi_i < \phi_j + \lambda_j\, g_j(x^i)$ for all $j \in B$ and $i \in E$ such that $x^i \neq x^j$,

(d) $\phi_j < \phi_i + \lambda_i\, g_i(x^j)$ for all $j \in B$ and $i \in E$ such that $x^i \neq x^j$,

(e) $\phi_i < \phi_j + \lambda_j\, g_j(x^i)$ for all $\{i, j\} \subseteq E$ such that $x^i \neq x^j$,

At the first execution of the algorithm we have $B = \varnothing$. After Step 6 has been executed once, B contains only the "equivalent" indices in E, i.e. indices $i \in E$ such that $x^i R_A^* x^\mu$. These elements are removed from I, such that at the second execution of Step 2, μ = MaxElement(I, R_A^*) cannot be in B. Indeed, after every execution of Step 6, μ at the next execution of Step 2 can never be in B.

Proof of (a): For all $i \in E$, we have $i \notin B$ because either $B = \varnothing$ by Step 1 or $B \cap I = \varnothing$

by Step 6. But if $\{i, j\} \subseteq I$ and $x^i = x^j$, then $\{i, j\} \subseteq E \subseteq I$, so $\{i, j\} \cap B = \varnothing$, hence the condition is always satisfied.

Proof of (b): If $x^i = x^j$, then either $\{i, j\} \subseteq E$ or $\{i, j\} \cap E = \varnothing$; furthermore, $\{i, j\} \cap B = \varnothing$. Then by Step 6 we have $\phi_i = \phi_j$.

Proof of (c): If $i \in E$, then $x^i R_A^* x^\mu$. Because μ is a maximal element of I, $x^\mu R_A^* x^i$. Since R^* satisfies SARQ, we must have $x^i = x^\mu$ for all $i \in E$. At the first execution of Step 6 we have $\phi_i = \lambda_i = 1$ for all $i \in E$. After the first execution of Step 6, we can either use the proof for (a) respectively (b), or we have that $\{\mu\} = E$. In the latter case, we have by Step 4,

$$\phi_i \leq \phi_j + \lambda_j g_j(x^i) - \varepsilon$$

and with $\varepsilon > 0$ we have

$$\phi_i < \phi_j + \lambda_j g_j(x^i).$$

Proof of (d): Note that at Step 5 we must have $g_i(x^j) > 0$ for all $j \in B$. If that were not the case, $x^i R_A x^j$ for some $j \in B$. But then i would have been moved to B before j was moved to B. Hence the division in Step 5 is well defined. We have

$$\lambda_i = \lambda_\mu \geq \frac{\phi_j - \phi_i + \varepsilon}{g_i(x^j)}.$$

Then

$$\lambda_i g_i(x^j) \geq \phi_j - \phi_i + \varepsilon$$

and with $\varepsilon > 0$ we have

$$\phi_j < \phi_i + \lambda^i g^i(x^j).$$

Proof of (e): If $\{i, j\} \subseteq E$, then $x^i R_A^* x^\mu$ and $x^\mu R_A^* x^i$ because μ is a maximal element of I. Because R_A^* satisfies SARQ, we must have $x^i = x^\mu$ for all $i \in E$, hence the condition is always satisfied. ■

A.4 *Testing whether a Point is in the Convex Hull of a Finite Set of Points*

Let $\Xi = \{\xi_i\}_{i=1}^{\ell}$ be a set of points in $\mathbb{R}^n$. Let ψ be a point in $\mathbb{R}^n$. To test whether ψ is contained in the convex hull of Ξ, one can set up a linear feasibility problem

$$\begin{aligned} \text{find} \quad & \psi \\ \text{such that} \quad & \psi = \sum_{i=1}^{\ell} \lambda_i \xi_i \\ & \sum_{i=1}^{n} \lambda_i = 1 \\ & \lambda_i \geq 0 \text{ for all } i = 1, \dots, \ell. \end{aligned} \tag{30}$$

Equivalently, one can set up a linear program. The problem (30) has a solution if and only if the following problem has no solution:

$$\begin{aligned} \text{find} \quad & \rho_0 \in \mathbb{R} \text{ and } \rho \in \mathbb{R}^n \\ \text{such that} \quad & \rho' \xi_i \leq \rho_0 \text{ for all } i = 1, \dots, \ell \\ & \rho' \psi > \rho_0 \end{aligned} \tag{31}$$

Then we can set up the linear program

$$\begin{aligned} h^* = \max \quad & \rho' \psi - \rho_0 \\ \text{subject to} \quad & \rho' \xi_i - \rho_0 \leq 0 \text{ for all } i = 1, \dots, \ell \\ & \rho' \psi - \rho_0 \leq 1 \end{aligned} \tag{32}$$

where the last condition is artificial so that the solution is bounded. The point ψ then is not in the convex hull (is not redundant) of Ξ if and only if h^* of the linear program (32) is strictly positive. See also Pardalos et al. (1995).

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