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Jan Heufer

Revealed Preference and the Number of Commodities

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Abstract

This work consists of two parts: First, it is shown that for a two-dimensional commodity space any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations. The result is stated as a pairwise version of Varian's Homothetic Axiom of Revealed Preference and is used to provide a simplified non-parametric test of homotheticity. In the second part a unifying proof technique is presented to show that the Weak Axiom of Revealed Preference (WARP) implies the Strong Axiom of Revealed Preference (SARP) for two commodities yet not for more commodities. It also shows that preference cycles can be of arbitrary length. While these results are already known, the proof here generalizes and unifies the existing ones insofar as it gives necessary and sufficient conditions for preference cycles to exist. It is then shown that in two dimensions the necessary condition cannot be fulfilled, whereas in more than two dimensions the sufficient conditions can always be met. The proof admits an intuitive understanding of the reason by giving a geometric interpretion of preference cycles as paths on indifference surfaces.

JEL Classification: C14, C60, D11, D12

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1. Introduction

For quite some time it had been an open question in economic theory whether the Weak Axiom of Revealed Preference (WARP) as introduced by Samuelson (1938) was actually sufficient to guarantee that a demand function maximizes a utility function. Houthakker (1950) defined an apparently stronger condition, the Strong Axiom of Revealed Preference (SARP) and showed that this condition was indeed sufficient. Arrow (1959), however, remarked that there was still no proof "that the Weak Axiom is not sufficient to ensure the desired result. The question is still open." Uzawa (1959) showed that the Weak Axiom combined with certain regularity conditions implies the Strong Axiom.[1] Meanwhile, Rose (1958) showed that the Weak Axiom implies the Strong Axiom for two commodities, extending a limited geometrical argument by Hicks (1965 [1956], pp. 52–54).[2]

Finally, Gale (1960) constructed a counterexample for the case of three commodities: WARP was satisfied, SARP was violated. This, essentially, settled the question. Kihlstrom, Mas-Colell, and Sonnenschein (1976) provided a theoretical argument which yields an infinite number of demand functions that satisfy WARP but not SARP. Peters and Wakker (1994, 1996) showed how to embed Gale's example in higher dimensions without relying on isomorphic extensions, i.e. with strictly positive demand for every commodity for suitable budgets. John (1997) showed that there is a simpler proof of their results.

Shafer (1977) showed that there exists a demand function for three commodities which violates SARP, but has no revealed preference cycles of length less or equal than any $k \geq 2$, which for $k = 2$ proves that WARP does not imply SARP.

This work consists of two parts. First, in Section 3 it is shown that Rose's (1958) result carries over to homothetic rationalization. That is, in the two-commodity case pairwise testing of observations is sufficient to test a set of observations on consumption choices for consistency with the maximization of a homothetic utility function. The result is stated as a pairwise version of Varian's (1983) Homothetic Axiom of Revealed Preference (HARP) and is used to provide a simplified nonparametric test of homotheticity. Second, Section 4 introduces a new approach to show (1) that WARP necessarily implies SARP for two commodities, (2) that WARP does not imply SARP for more than two commodities, (3) that for more than two commodities there can be preference cycles of arbitrary finite length, (4) how to construct examples for the preceding two points. The approach here unifies and generalizes the proofs of Rose (1958), Gale (1960), and Peters and Wakker (1994) insofar that necessary and sufficient conditions for cycle length greater than two are given. It is shown that in two dimensions the necessary conditions cannot hold, whereas in more than two dimensions the sufficient conditions can be sat-

[1] Samuelson is said to have expressed the view that these regularity conditions "are perhaps integrability conditions in disguise" (Gale 1960), and Kihlstrom, Mas-Colell, and Sonnenschein (1976) commented that "it looks very much like the *strong axiom* itself".

[2] Banerjee and Murphy (2006) use the result to provide a simplified nonparametric test of Varian's (1982) Generalized Axiom of Revealed Preference (GARP).

isfied. The proof admits an intuitive understanding of the reason by giving a geometric interpretion of preference cycles as paths on indifference surfaces.

2. Preparations

2.1. Basic Definition

Let $X = \mathbb{R}^\ell_+$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities.[3] The price space is $P = \mathbb{R}^\ell_{++}$, and the space of price-income vectors is $P \times \mathbb{R}_{++}$. Consumers choose bundles $x^i = (x^i_1, \ldots, x^i_\ell)' \in X$ when facing a price vector $p^i = (p^i_1, \ldots, p^i_\ell) \in P$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(p^i, w^i) = \{x \in X : p^i x^i \leq w^i\}$. To prevent any misconception about the generality of the approach in Section 4, the demand for each commodity $j \leq \ell$ is assumed to be strictly positive for a suitable budget.[4] Demand is exhaustive, i.e. $p^i x^i = w^i$. Denote the upper bound of the budget set B as $\bar{B} = \{x \in X : px = w\}$, so $x^i \in \bar{B}^i$. Prices are normalized by the level of expenditure at each observation, so that $w^i = p^i x^i = 1$ for all i. A set of n observations can then be denoted as $S = \{(x^i, p^i)\}_{i=1}^n$.

2.2. Revealed Preference

Let $R, R^*, R^s \subseteq X \times X$ be binary relations on X. An observation x^i is *directly revealed preferred* to x, written $x^i R x$, if $p^i x^i \geq p^i x$. It is *revealed preferred* to x, written $x^i R^* x$, if either $x^i R x$ or for some sequence of bundles $(x^j, x^k, \ldots, x^m)$ it is the case that $x^i R x^j$, $x^j R x^k$, …, $x^m R x$. In this case R^* is the *transitive closure* of the relation R, i.e. $R^* = \bigcup_{i=1,2,\ldots} R^i$. An observation x^i is *strictly directly revealed preferred* to a bundle x, written $x^i R^s x$, if and only if $p^i x^i > p^i x$.

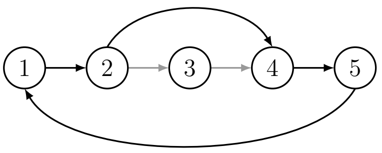
The data set S satisfies the WARP if $x^i R x^j$, $x^i \neq x^j$, does not imply $x^j R x^i$. The data set S satisfies the SARP if $x^i R^* x^j$, $x^i \neq x^j$, does not imply $x^j R x^i$.

The set of bundles that are revealed preferred to a certain bundle x^0 (which does not have to be an observed choice) is given by the convex monotonic hull of all choices revealed preferred to x^0, i.e. $RP(x^0) =$ convex hull of $\{x \in X : x \geq x^i$ such that $x^i R^* x^0\}$. See Varian (1982) and Knoblauch (1992). The convex monotonic hull of a set of points $\{x^i\}$ will be denoted as CMHull$(\{x^i\}) =$ convex hull of $\{x \in X : x \geq x^i\}$.

The set of observations S can be interpreted as an unweighted directed graph (digraph), i.e. a pair $G = (V, A)$ where V is the set of nodes (the observations) and A is the set of directed edges or arcs (the directly revealed preference relations). An arc $a_{ij} = \{x^i, x^j\}$ is directed from x^i to x^j and is an element of A if and only if $x^i R x^j$.

[3] Notation: $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell : x \geqq 0\}$, $\mathbb{R}^\ell_{++} = \{x \in \mathbb{R}^\ell : x > 0\}$, where " $x \geqq y$ " means " $x_i \geq y_i$ for all i ", " $x \geq y$ " means " $x \geqq y$ and $x \neq y$ ", and " $x > y$ " means " $x_i > y_i$ for all i ". Note the convention to use subscripts to denote scalars or vector components and superscripts to index bundles.

[4] Note that an example for three commodities obviously implies that there exists examples for more than three commodities. But without relying on isomorphic extensions, embedding the example in higher dimensions is not trivially.



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 1: Left: The observations can be interpreted as nodes of a digraph. The shortest cycle includes nodes 1, 2, 4 and 5. Right: The Boolean adjacency matrix of the graph.

The graph can then be represented by a Boolean adjacency matrix $M = \{m_{ij}\}$ where $m_{ij} = 1$ if $x^i R x^j$ and $m_{ij} = 0$ otherwise.[5]

An ordered set $\{(x^i, p^i)\}_{i=1}^k$ of k observations forms a *cycle of length* k if $p^i x^{i+1} \leq p^i x^i$ and $x^i \neq x^{i+1}$ for $i = 1, \ldots, k, k+1 \bmod k+1$, i.e. if x^i is indirectly revealed preferred to itself via the chain of observations $\{(x^i, p^i)\}_{i=1}^k$. A set $\{(x^i, p^i)\}_{i=1}^k$ forms a *cycle of irreducible length* k if it forms a cycle of length k and there is no shorter cycle (with a smaller k) by which x^i is indirectly revealed preferred to itself. As an illustration, suppose there is a set of observations $\{(x^i, p^i)\}_{i=1}^5$ such that $x^1 R x^2$, …, $x^4 R x^5$. Suppose that also $x^5 R x^1$ and $x^2 R x^3$, but there are no other directly revealed preference relations. Then by $x^1 R^* x^5$ and $x^5 R x^1$ there is a preference cycle. The irreducible length of the shortest cycle in that data is four. See Figure 1.

Obviously, WARP implies the absence of cycles of irreducible length two, whereas SARP implies the absence of cycles of arbitrary irreducible length.

2.3. Revealed Homothetic Preference

A utility function is *homothetic* if it is a positive monotonic transformation of a utility function that is homogenous of degree 1.

The set S satisfies the Homothetic Axiom of Revealed Preference (HARP) if for all distinct choices of indices $(i, \ldots, \ell)$

$$(p^i x^j)(p^j x^k) \cdots (p^m x^i) \geq w^i w^j w^k \cdots w^m.$$

Theorem 1 (Varian 1983) *The following conditions are equivalent:*
(1) there exists a concave, monotonic, continuous, non-satiated, homothetic utility function that rationalizes the data;
(2) the data satisfy HARP.

Following Varian (1983) and Knoblauch (1993), define a scalar $t^{i,j}$ for all i and j by

$$t^{i,j} = \min \left\{ \left(\frac{p^i x^k}{w^i} \right) \left(\frac{p^k z^\ell}{w^k} \right) \cdots \left(\frac{p^m z^j}{w^m} \right) \right\},$$

[5] One can then use Warshall's algorithm (Warshall 1962) to compute the transitive closure of the binary relation R. In the context of revealed preference theory this has first been pointed out and used by Varian (1982).

where the minimum is over all distinct choices of indices $k, \ell, \ldots, m$. Let $t^{i,i} = 1$. Then $t^{i,j}x^i$ is *homothetically revealed preferred* to x^j, written $t^{i,j}x^iHx^j$. Note that $t = t^{i,j}$ is the smallest scalar for which tx^iHx^j, so that $t^{i,j}x^i$ is a vertex on the set of bundles that are homothetically revealed preferred to x^j (see Knoblauch (1993)). HARP is then equivalent to $\neg(x^iR^st^{j,i}x^j)$ for all i and j, where $\neg$ means "not true".

3. Homothetic Preference and Two-Commoditity Choice

3.1. Theory

When the consumption space is two-dimensional, the budgets can be ranked by the price ratio. In this section vectors are given in boldface; consumption bundles are denoted by $\mathbf{z}$. Let $\mathbf{z}^i = (x^i, y^i)'$ and choose good x as the numeraire. Then $\mathbf{p}^i = (1, q^i)$, where q^i is the relative price of good y. Let the income w^i be redefined appropriately. Without loss of generality, let the data S be ordered by q such that $q^i \geq q^{i+1}$. If there are observations with the same q, let them be ordered such that $y^i/x^i \leq y^{i+1}/x^{i+1}$.

It is a well known fact that homotheticity implies that income expansion paths are straight lines through the origin. It is easy to show that the slope of the expansion path, y/x, must increase as the relative price of y decreases: In case of homotheticity, $(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq (\mathbf{p}^i\mathbf{z}^i)(\mathbf{p}^j\mathbf{z}^j)$. That is equivalent to $(q^i - q^j)(x^iy^j - y^ix^j) \geq 0$. If $i < j$, $(q^i - q^j) \geq 0$, so it must be that $(x^iy^j - y^ix^j) \geq 0$. Thus $y^i/x^i \leq y^j/x^j$, and analogously for $i > j$. This is obviously a necessary condition for homotheticity, but it is not obvious that it is also sufficient.

Definition The data satisfies the Pairwise Homothetic Axiom of Revealed Preference (PHARP) if for all distinct choices of indices i, j

$$(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq w^iw^j.$$

Theorem 2 *If the commodity space is two-dimensional, the following conditions are equivalent:*

(1) there exists a concave, monotonic, continuous, non-satiated, homothetic utility function that rationalizes the data;

(2) the data satisfy HARP;

(3) the data satisfy PHARP.

Proof. For (1) $\Leftrightarrow$ (2), see Varian (1983). It is obvious that (2) $\Rightarrow$ (3). We will show that (3) $\Rightarrow$ (2).

The following Lemma will be helpful:

Lemma 1 *Define a scalar $\theta^{i,j}$, where i and j are indices, by*

$$\theta^{i,j} = \prod_{k=j}^{i-1} \frac{\mathbf{p}^{k+1}\mathbf{z}^k}{\mathbf{p}^{k+1}\mathbf{z}^{k+1}} \text{ if } i > j \quad \text{and} \quad \theta^{i,j} = \prod_{k=i}^{j-1} \frac{\mathbf{p}^k\mathbf{z}^{k+1}}{\mathbf{p}^k\mathbf{z}^k} \text{ if } i < j.$$

Then if the commodity space is two-dimensional and the data set satisfies PHARP, $\theta^{i,j} = t^{i,j}$.

Proof of Lemma 1: See appendix.

Choose a $\mathbf{z}^0$ without loss of generality, i.e. assign indices such that q^0 is the highest, the lowest, or somewhere between the highest and the lowest relative price. Then $\neg(\mathbf{z}^0 R^s \theta^{1,0} \mathbf{z}^1)$ if PHARP is satisfied. We need to show that this implies $\neg(\mathbf{z}^0 R^s \theta^{n,0} \mathbf{z}^n)$ for all $n > 0$.

Suppose $\neg(\mathbf{z}^0 R^s \theta^{n,0} \mathbf{z}^n)$. Then

$$\begin{aligned}
& \mathbf{p}^0\mathbf{z}^0 \leq (\mathbf{p}^0\mathbf{z}^n)\theta^{n,0} \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,0} = (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n}\theta^{n,0} \\
\Leftrightarrow \quad & \mathbf{p}^0\mathbf{z}^n \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n} \;\Leftrightarrow\; (\mathbf{p}^0\mathbf{z}^n)(\mathbf{p}^{n+1}\mathbf{z}^{n+1}) \leq (\mathbf{p}^0\mathbf{z}^{n+1})(\mathbf{p}^{n+1}\mathbf{z}^n) \\
\Leftrightarrow \quad & (q^0 - q^{n+1})(x^{n+1}y^n - x^n y^{n+1}) \leq 0.
\end{aligned}$$

Is is easy to see that the last line is true because if $n > 0$ and PHARP is satisfied the first term on the left hand side is positive while the second term is negative. Analogously for $n < 0$. This proves that $\neg(\mathbf{z}^0 R^s \theta^{1,0} \mathbf{z}^1)$ implies $\neg(\mathbf{z}^0 R^s \theta^{n,0} \mathbf{z}^n)$ for arbitrary $\mathbf{z}^0$. So PHARP implies HARP. □

Remark The theorem can also be extended to fit the stronger notion of homotheticity as defined by Liu and Wong (2000).

3.2. Testing for Homotheticity in the Two-Commodity Case

A quick way to test if a set of consumption data satisfies homotheticity is to compute the matrix $\mathbf{C} = \{c_{i,j}\}$, where

$$c_{i,j} = \frac{\mathbf{p}^i\mathbf{z}^j}{w^i}\frac{\mathbf{p}^j\mathbf{z}^i}{w^j}$$

and check if any element of $\mathbf{C}$ is less than 1. If there is a unique ordering of the relative prices, it is sufficient to only compute and check the subdiagonal for the ordered data. Varian's (1983) nonparametric test for homotheticity requires the use of algorithms that can detect negative weight cycles, like the Warshall algorithm. For the two-commodity case no such algorithm is needed.

4. Preference Cycles and the Number of Commodities

4.1. Theory

Obviously any hyperplane that has an interior point of a convex polytope on one side will also have at least one vertex of the polytope on the same side.

This can be interpreted in the context of revealed preference: There can be observations that are strictly in a set $RP(x^0)$ and hence are redundant. If an observation x^i is directly preferred to such an interior point, the budget hyperplane $\bar{B}^i$ has to intersect the set $RP(x^0)$. Then $\bar{B}^i$ has at least one vertex of $RP(x^0)$ on its "left" side, so x^i is

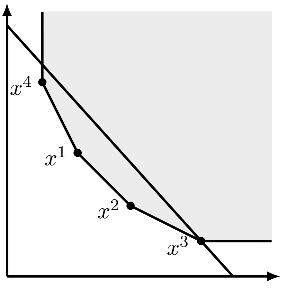


Figure 2: Obviously x^3 cannot be directly revealed preferred to x^4 without also being directly revealed preferred to x^1 and x^2.

also directly revealed preferred to at least one other vertex of $RP(x^0)$. This leads to Proposition 1.

Proposition 1 *Suppose $T = \{(x^i, p^i)\}_{i=1}^k$ is an ordered set of observations that forms a cycle of irreducible length k such that x^1Rx^2, x^2Rx^3, $\ldots$, $x^{k-1}Rx^k$, x^kRx^1. Then all of the observations in the cycle have to be distinct and non-redundant vertices of* CMHull(T)*, and the line segments connecting two observations of which one observation is directly revealed preferred to the other have to be edges on the boundary of* CMHull(T).

Proof of Proposition 1: See appendix.

Corollary 1 *WARP implies SARP for two commodities.*

Proof of Corollary 1. Suppose there is a cycle x^1Rx^2, x^2Rx^3, $\ldots$, $x^{k-1}Rx^k$, x^kRx^1 of length k. By Proposition 1 the two edges connecting x^{k-1} with x^k and x^k with x^1 have to be on the boundary of the convex monotonic hull of all observations in the cycle. Because in a two-dimensional convex hull any vertex has only two edges, x^{k-1} and x^1 have to be either equal or on different sides of x^k. If $x^{k-1} = x^1$, there is a cycle of length two. If $x^{k-1} \neq x^1$, at one point an edge connecting some x^i with x^{i+1} needs to cut through the convex monotonic hull. Therefore there cannot be a cycle of irreducible length greater than two. See Figure 2. □

Corollary 2 *The shortest revealed preference cycle has to be of length two for two commodities. This follows directly from Corollary 1.*

Remark In contrast to Gale's (1960) proof, Proposition 1 gives necessary conditions for the existence of preference cycles of length $k > 2$. It is then shown that these condition cannot be met in two dimensions.

Proposition 2 *Suppose $T' = \{x^i\}_{i=1}^k$ is a set of bundles such that all $x^i \in T'$ are distinct and non-redundant vertices on* CMHull(T'). *Then if there are non-intersecting line segments connecting all $x^{i \bmod (k+1)}$ with $x^{i+1 \bmod (k+1)}$ for all $x^i \in T'$ such that* *these line segments are edges of* CMHull(T')*, there exists a set of price vectors* $\{p^i\}_{i=1}^k$, $p^i \in P\ \forall i \leq k$*, such that* $\{(x^i, p^i)\}_{i=1}^k$ *forms a cycle of irreducible length* k*.*

Proof of Proposition 2. By the supporting hyperplane theorem there exists a hyperplane $H(p) = \{x \in X : px = 1\}$ such that $x^i, x^{i+1} \in H(p)$ and $x^j \notin H(p)$ for all $j \neq i, i+1$. Let p be the price vector at which x^{i+1} was chosen, so that $\bar{B}^{i+1} = H(p)$. Clearly, $x^{i+1}Rx^i$ and $\neg(x^{i+1}Rx^j)$, i.e. $x^j \notin B^{i+1}\ \forall j \notin \{i, i+1\}, j \leq k$. □

Remark The conditions in Proposition 2 extend the conditions given in Proposition 1. The combination is sufficient for the existence of preference cycles of arbitrary length.

Proposition 3 *For* $\ell > 2$ *there always exists a set of bundles* $T' = \{x^i\}_{i=1}^k$ *such that all* $x^i \in T'$ *are distinct and non-redundant vertices on* CMHull(T') *and there are non-intersecting edges of* CMHull(T') *that connect all* $x^{i \bmod (k+1)}$ *with* $x^{i+1 \bmod (k+1)}$ *for all* $x^i \in T'$*.*

Proof of Proposition 3. A simple way to find a set of bundles T' that satisfies the conditions is to take a set of k distinct points from the intersection of an indifference hypersurface of a concave utility function and a hyperplane $H(q) = \{x \in X : qx = 1\}$. The intersection of two convex sets is again convex, so there are no interior or redundant points in the convex hull of the stereographic projection of all $x^i \in T'$ on a projective plane. Obviously the edges of that convex hull do not intersect and connect all $x^{i \bmod (k+1)}$ with $x^{i+1 \bmod (k+1)}$ for all projected points. (See also Figures 3, 4, and 6.) □

Corollary 3 *For more than three commodities the shortest revealed preference cycle can be of arbitrary finite irreducible length. This follows directly from Proposition 3 which holds for any* k*.*

Corollary 4 *For more than three commodities WARP does not imply SARP. This follows directly from Corollary 3.*

Remark In the final step it was shown that the sufficiency conditions given in Proposition 2 can be met in three or more dimensions.

4.2. Intuition

A graph $G(V, A)$ as defined in Section 2 that represents a preference cycle of irreducible length k is always a planar graph, i.e. a graph that can be drawn in the plane so that no edges intersect. Therefore $G(V, A)$ can always be embedded in an indifference surface of dimension three or more in the sense that every $v_i \in V$ is associated with a point on the surface, and every arc $a_{ij} \in A$ is associated with an edge. It cannot, however, be embedded in a two-dimensional indifference curve. That is to say, one cannot "extract" a preference cycle longer than two from a two-dimensional commodity space, whereas higher dimensions allow this, as shown in Proposition 3 and Figures 3 and 4.

For a similar intuition consider this: Just as there is only one distinct path on a circle (a closed curve) that connects a certain point on the circle with itself, there is no such

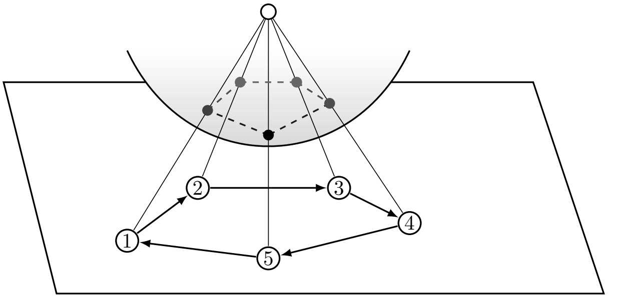


Figure 3: A graph that represents a preference cycle can be "extracted" from a three-dimensional indifference surface.

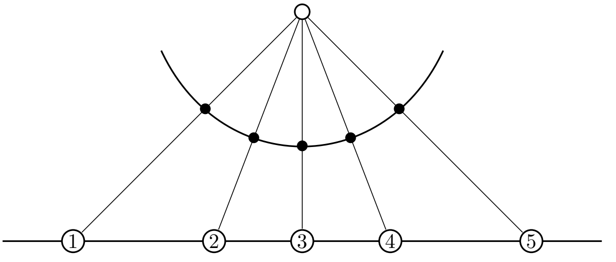


Figure 4: No graph that represents a preference cycle can be "extracted" from a two-dimensional indifference curve.

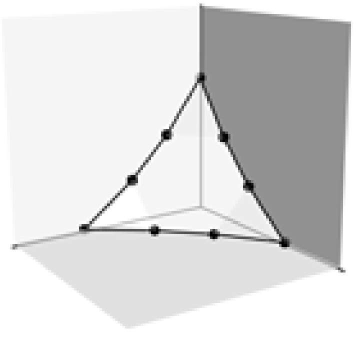


Figure 5: Gale's example.

path on an indifference curve (which is not closed). And just as there are infinitely many distinct paths on a sphere that connect a certain point on the sphere with itself, there are infinitely many paths on an indifference surface of a dimension greater than two.

Figure 5 shows the nine bundles in Gale's (1960) example and the convex monotonic hull of these points. Quite obviously it is possible to draw a line on the boundary of the hull that connects all points without intersecting itself.

Figure 6 shows how one can easily construct examples of preference cycles of arbitrary length in the three-dimensional commodity space. By Proposition 2, there exist price vectors such that each edge that connects two points is a line segment in the budget hyperplane at which one of the points was chosen, so that one of the points is directly revealed preferred to the other. Note that when one tries to use this method to construct a preference cycle in two dimensions, one obtains exactly two points – which is the maximal cycle length in two dimensions.

5. Conclusion

In the first part of this paper it was shown that for two-dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations. The result exploits the possibility of ranking budgets by their slope, which is only possible when the consumption space is two-dimensional. A straightforward application is to simplify the nonparametric test for homotheticity, so that the use of Warshall's algorithm can be avoided.

Another possible application is to use the result for a nonparametric test of homotheticity for discrete budget sets. By Lemma 1 it is possible to use Knoblauch's (1993) method of recovering homothetic preferences which are implicit in a set of consumption data even if homotheticity is violated. The resulting homothetic bounds on the indiffer-

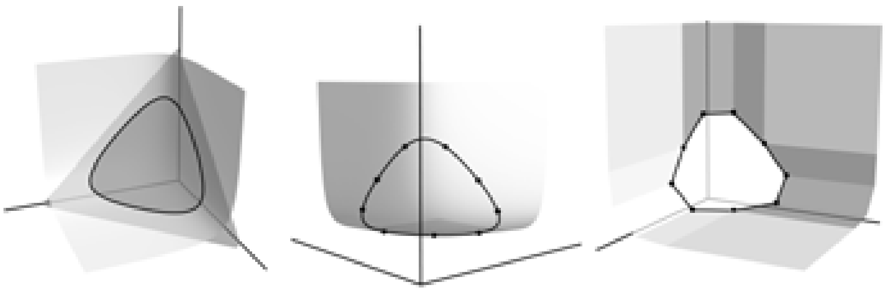


Figure 6: A simple example for the construction of a cycle. Left: The indifference surface of the utility function $u(x_1, x_2, x_3) = x_1^{1/3} x_2^{1/3} x_3^{1/3}$ for $\bar{u} = 1$ and the plane $H = \{x \in \mathbb{R}^3_+ : 1/4(x_1 + x_2 + x_3) = 1\}$. Center: A set of points on the intersection of the indifference curve and the hyperplane. Right: The convex monotonic hull of the points.

ence curve would then intersect at least some of the budget lines. One could then check if there have been bundles available on or below such a budget line which are within the homothetic revealed preferred set. If not, the conclusion would be that the consumer would not have violated homotheticity if the budget sets were continuous. That is done in Heufer (2007).

In the second part of this paper, a new approach to proof some established results was presented: WARP necessarily implies SARP for two commodities, WARP does not imply SARP for more than two commodities, and for more than two commodities there can be preference cycles of arbitrary finite length. It was also shown how to construct examples. The proof unifies the existing proofs by giving necessary and sufficient conditions for the existence of preference cycles of a length greater than two. In two dimensions the necessary conditions cannot be fulfilled, whereas in three or more dimensions the sufficient conditions can be met.

It was first shown that all of the observations in a preference cycle have to be distinct vertices and the line segments connecting two observations of which one observation is directly revealed preferred to the other have to be edges on the boundary of the convex monotonic hull of all bundles involved in the cycle. This was used to show that WARP does indeed imply SARP for two commodities.

Next it was shown that if there are non-intersecting line segments connecting each bundle with the bundle it is directly revealed preferred to such that these line segments are edges of the convex monotonic hull of all bundles, one can find a set of corresponding price vectors such that the observations form a cycle of irreducible length $k > 2$. It was also shown that such sets of bundles exist and can be obtained by taking points from the intersection of two convex sets. This was used to show that WARP does not imply SARP for more than two commodities.

An interesting aspect about the proofs is that they give a nice intuition about the reason why WARP implies SARP for two commodities yet not for more commodities, by interpreting preference cycles as paths on indifference surfaces.

A. Appendix

Proof of Lemma 1. Choose a $\mathbf{z}^0$ without loss of generality . It is first shown that $\theta^{1,0} = t^{1,0}$. Remember that the observations are ordered such that $q^i \geq q^{i+1}$.

$$\theta^{1,0} = \frac{\mathbf{p}^1\mathbf{z}^0}{\mathbf{p}^1\mathbf{z}^1} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1}\frac{\mathbf{p}^i\mathbf{z}^0}{\mathbf{p}^i\mathbf{z}^i} \Leftrightarrow (\mathbf{p}^1\mathbf{z}^0)(\mathbf{p}^i\mathbf{z}^i) - (\mathbf{p}^1\mathbf{z}^i)(\mathbf{p}^i\mathbf{z}^0) \leq 0$$
$$\Leftrightarrow \quad (q^1 - q^i)(x^i y^0 - x^0 y^i) \leq 0.$$

The last line is true because if $i > 1$, the first term is positive and the second term is negative, and vice versa if $i < 1$. Now suppose

$$\theta^{1,0} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k}$$

for a sequence $i, \ldots, k$ of length n. Then $\theta^{1,0}$ is also less than or equal to a sequence $i, \ldots, \ell$ of length $n+1$ because

$$\frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^\ell}{\mathbf{p}^k\mathbf{z}^k}\frac{\mathbf{p}^\ell\mathbf{z}^0}{\mathbf{p}^\ell\mathbf{z}^\ell}$$
$$\Leftrightarrow \quad (\mathbf{p}^k\mathbf{z}^0)(\mathbf{p}^\ell\mathbf{z}^\ell) \leq (\mathbf{p}^k\mathbf{z}^\ell)(\mathbf{p}^\ell\mathbf{z}^0),$$

where the last line is true for similar reasons as above. So $\theta^{1,0} = t^{1,0}$.

It is now possible to show that $\theta^{n,0} = t^{n,0}$ implies $\theta^{n+1,0} = t^{n+1,0}$, which concludes the proof by induction: Write $\theta^{n+1,0}$ as $\theta^{n+1,n}\theta^{n,0}$ and note that $\theta^{n+1,n} = (\mathbf{p}^{n+1}\mathbf{z}^n)/(\mathbf{p}^{n+1}\mathbf{z}^{n+1})$. Then is is to be shown that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\theta^{n,0} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)},$$

for sequences of $i, \ldots, k$ of arbitrary length. By assumption,

$$\theta^{n,0} \leq \frac{(\mathbf{p}^n\mathbf{z}^i)}{(\mathbf{p}^n\mathbf{z}^n)}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)}.$$

It is then sufficient that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^n\mathbf{z}^n)}{(\mathbf{p}^n\mathbf{z}^i)}$$

holds, which is true if $n > 0$. The proof works analogously for $n < 0$. □

Proof of Proposition 1. Suppose $x^i \in T$ is not a vertex on the convex monotonic hull of T. Then x^{i-1} is directly revealed preferred to some x^j, $j \notin \{i-1, i\}$. If $j < i-1$ there exists a sequence $x^j R x^{j+1}$, $x^{j+1} R x^{j+2}$, …, $x^{i-2} R x^{i-1}$, $x^{i-1} R x^j$ of length $i - j < k$ which constitutes a preference cycle. If $j > i$ there exists a sequence $x^{i-1} R x^j$, $x^j R x^{j+1}$, …, $x^{k-1} R x^k$, $x^k R x^1$, …, $x^{j-1} R x^j$ of length $i - j + k < k$ which constitutes a preferences cycle.

Suppose that the line segment connecting x^{i-1} and x^i is not an edge of the convex monotonic hull. Then x^{i-1} also has to be directly revealed preferred to some x^j which is a vertex that causes the line to be strictly in the convex monotonic hull. Obviously this causes the cycle to be shorter than k by the same token as above.

Suppose $x^i \in T$ is a redundant vertex on the boundary so that $x^i = \lambda x^j + (1-\lambda)x^k$, $0 < \lambda < 1$, for some $x^j, x^k \in T$. Then x^i is on a line segment connecting x^j and x^k such that either (1) $x^j R x^k$ or $x^k R x^j$, or (2) $x^j R x^i \wedge x^i R x^k$ or $x^k R x^i \wedge x^i R x^j$. Case 1 implies $x^j R x^i$ or $x^k R x^i$ respectively. Case 2 implies $x^j R x^k$ or $x^k R x^j$ respectively. In either case, one bundle in $\{x^i, x^j, x^k\}$ is directly revealed preferred to two other bundles, which reduces the length of the preference cycle by the same token as above.

Suppose $x^i \in T$ is a redundant vertex on the boundary because it is a point on the monotonic extension of the convex hull of all bundles in T, so that $x^i \geq x^j$ for some x^j. Obviously any bundle directly revealed preferred to x^i will also be directly revealed preferred to x^j, which reduces the length of the preference cycle by the same token as above. $\square$

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